1. A
2. B
3. C
4. A
5. C
6. A
7. B
8. C
9. A
10. E
11. A
12. E
13. E
14. C
15. E
16. B
17. D
18. D
19. C
20. B
21. B
22. A
23. B
24. D
25. C
26. D
27. B
28. B
29. B
30. D

1. A Use the divisibility rule for 11. 2024 ends up being a multiple. Alternatively, just long divide.
2. B Factoring, $76 = 2^2 \cdot 19$. So, the sum of the positive divisors of 76 is:
$$(2^0 + 2^1 + 2^2)(19^0 + 19^1) = 140$$
However, we want the sum of the proper divisors of 76, so the answer is actually $140 - 76 = 64$.
3. C It is possible to factor the numbers, but it's ugly: $8398 = 2 \cdot 13 \cdot 17 \cdot 19$ and $1729 = 7 \cdot 13 \cdot 19$. Instead, it's much better to use the Euclidean Algorithm:
$$\begin{aligned}\gcd(8398, 1729) &= \gcd(1482, 1729)\\ &= \gcd(1482, 247)\\ &= \gcd(0, 247)\end{aligned}$$
giving us answer of 247.
4. A We use the well-known properties:
   1) $\gcd(na, nb) = n \cdot \gcd(a, b)$
   2) $\text{lcm}(a, b) \cdot \gcd(a, b) = ab$

   Then, noting that $420 = 4(105)$ and $728 = 4(182)$, we find:
$$\begin{aligned}\text{lcm}(105,182) \cdot \gcd(728,420) &= \text{lcm}(105,182) \cdot 4 \cdot \gcd(182,105)\\ &= 4 \cdot \text{lcm}(105,182) \cdot \gcd(105,182)\\ &= 4 \cdot (105 \cdot 182) = 76440.\end{aligned}$$
5. C A positive integer has an odd number of factors if and only if it is a perfect square. As $44^2 = 1936$ and $45^2 = 2025$, there are 44 perfect squares less than 2024. So, there are $2023 - 44 = 1979$ positive integers less than 2024 with an even number of factors.
6. A It is well-known that there are 25 primes less than 100. Thus, it remains to check the number from 101-119, inclusive. It turns out that the primes in this range are 101, 103, 107, 109, and 113. Hence, our answer is $25 + 5 = 30$.
7. B The sum of digits of 38547 is $3 + 8 + 5 + 4 + 7 = 27$, so the number is divisible by 3. In fact, its prime factorization is $3^2 \cdot 4283$. The other numbers can be computer verified as primes.
8. C Let $\varphi(n)$ be Euler's totient function. Since $\gcd(p, q) = \gcd(p, q + kp)$, the number of integers in [1, 2002] that are relatively prime to 1001 is equal to $2 \cdot \varphi(1001)$. Also, the number of integers in [2003, 2024] that are relatively prime to 1001 is the same as in [1, 22].
Thus, we first compute $\varphi(1001)$. Prime factoring, we get $1001 = 7 \cdot 11 \cdot 13$. Then
$$\varphi(1001) = 1001 \cdot \left(1 - \frac{1}{7}\right)\left(1 - \frac{1}{11}\right)\left(1 - \frac{1}{13}\right) = 720$$
Now, for the integers in [1, 22] that are relatively prime to 1001, we can just remove the multiples of 7, 11, and 13. We end up removing 7, 14, 21, 11, 22, and 13, so there are $22 - 6 = 16$ numbers in the interval that work. Hence, we have in total, $2 \cdot 720 + 16 = 1456$ integers.
9. A We can just write out the elements of the Fibonacci sequence, starting from F_1, modulo 21 until we find a pattern
$$1,1,2,3,5,8,13,0,13,13,5,18,2,20,1,0,1,1,2,3, ...$$

It's clear the sequence cycles, and carefully counting, we see $F_n \equiv F_{n+16}\ (mod\ 21)$. Then, we compute

$$F_{2024} \equiv F_{2024\ (\text{mod } 16)} (\text{mod } 21) \Rightarrow F_{2024} \equiv F_8 \equiv 0\ (mod\ 21)$$

10. E You can't apply Chicken McNugget Theorem because the two numbers aren't relatively prime. Any number that isn't a multiple of 7 is unachievable, so there is no upper bound.

11. A Consider everything modulo 6. We know $6 \equiv 0, 28 \equiv 4, 91 \equiv 1$ (mod 6). Using this, we can determine, for each residue modulo 6, the smallest number of bowls that can be bought exactly. After a little experimentation, we find

$$0\ (\text{mod } 6): 0$$
$$1\ (\text{mod } 6): 91$$
$$2\ (\text{mod } 6): 2(28) = 56$$
$$3\ (\text{mod } 6): 2(28) + 91 = 147$$
$$4\ (\text{mod } 6): 28$$
$$5\ (\text{mod } 6): 28 + 91 = 117$$

The largest number we find is 147, so we subtract 6 because we want the largest unachievable number. Thus, our answer is 141.

12. E We use the fact that

$$\binom{127}{39} = \frac{127!}{39! \cdot 88!}$$

Then applying Legendre's formula to find the greatest power of 2 that divides 127!, 39!, and 88!, we find:

$$127!: 63 + 31 + 15 + 7 + 3 + 1 = 120$$
$$39!: 19 + 9 + 4 + 2 + 1 = 35$$
$$88!: 44 + 22 + 11 + 5 + 2 + 1 = 85$$

Subtracting, we find that there are $120 - 35 - 85 = 0$ powers of 2 in $\binom{511}{138}$.

13. E Let n be the number of cards Jeffrey bought. It is given that n is 1 (mod 6), 11 (mod 14), and 9 (mod 26). We solve this congruence using Chinese Remainder Theorem. However, instead of just solving each pair of congruences, which is a cumbersome process, we do something a bit cleverer. Writing the congruences as negatives then manipulating, we get

$$1 \equiv -5 \equiv -17\ (\text{mod } 6)$$
$$11 \equiv -3 \equiv -17\ (\text{mod } 14)$$
$$9 \equiv -17\ (\text{mod } 26)$$

Then, the solution to our congruence is

$$-17\ (\text{mod lcm}(6{,}14{,}26)) \longrightarrow -17\ (\text{mod } 546) \equiv 529\ (\text{mod } 546)$$

Clearly, the smallest possible number of cards Jeffrey could have bought is 529.

14. C It is well known that the summation $\sum_{d|n} \varphi(d)$ is equal to n for all positive integers n. However, $30 = 2 \cdot 3 \cdot 5$ only has $(1+1)(1+1)(1+1) = 8$ positive factors, so the sum is also reasonable to do by hand. It becomes

$$\sum_{d|30, d \in Z^+} \varphi(d) = \varphi(1) + \varphi(2) + \varphi(3) + \varphi(5) + \varphi(6) + \varphi(10) + \varphi(15) + \varphi(30)$$
$$= 1 + 1 + 2 + 4 + 2 + 4 + 8 + 8 = 30$$

15. E For any integer $n \geq 1$, we have $\varphi(3^n) = \frac{2}{3} \cdot 3^n = 2 \cdot 3^{n-1}$. Then our summation becomes

$$\varphi(3^0) + \sum_{n=1}^{8} 2 \cdot 3^{n-1} = \varphi(1) + 2 \cdot \sum_{n=0}^{7} 3^n$$
$$= 1 + 2 \cdot \frac{3^8 - 1}{3 - 1}$$
$$= 1 + 3^8 - 1 = 6561$$

16. B We use Euler's totient theorem to simplify 7^{2024} (mod 1000). Factoring, we get $1000 = 2^3 \cdot 5^3$, so we know

$$\varphi(1000) = 1000 \cdot \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right) = 400$$

Then we can manipulate

$$7^{2024} \equiv (7^{400})^5 \cdot 7^{24} \equiv 7^{24} \text{ (mod 1000)}$$

Using the fact that $7^4 = 2401 \equiv 401$ (mod 1000), we can compute our desired quantity

$$7^8 \equiv 401^2 \equiv 801 \text{ (mod 1000)}$$
$$7^{16} \equiv 801^2 \equiv 601 \text{ (mod 1000)}$$
$$7^{24} \equiv 7^8 \cdot 7^{16} \equiv 801 \cdot 601 \equiv 401 \text{ (mod 1000)}$$

17. D Long dividing, the fraction becomes $n + 7 + \frac{120}{n-11}$. Clearly, n is of the form $d + 11$, where d is an integer such that $d \mid 120$. So, we plug in $n = d + 11$ and manipulate

$$\sum_{n \in N} \left(n + 7 + \frac{120}{n - 11}\right) = \sum_{d|120,\, d \in Z} \left(d + 18 + \frac{120}{d}\right)$$
$$= \sum_{d|120,\, d \in Z} d + \sum_{d|120,\, d \in Z} (18) + \sum_{d|120,\, d \in Z} \frac{120}{d}$$

The values of d and $\frac{120}{d}$ will cycle through all the positive and negative divisors of 120, so the two summations are equal to 0. Since $120 = 2^3 \cdot 3 \cdot 5$ has $(3 + 1)(1 + 1)(1 + 1) = 16$ positive divisors, there are 32 possible values of d. Then our summation becomes

$$\sum_{d|120,\, d \in Z} \left(d + 18 + \frac{120}{d}\right) = 0 + \sum_{d|120,\, d \in Z} 18 + 0 = 32(18) = 576$$

18. D Let $100a + 10b + c = abc_{10}$. We know that 16! is a multiple of 7,11, and 13, so it must be a multiple of 1001. Note that $10^3 = 1000 \equiv -1$ (mod 1001). With this, we can derive

$$16! = 20 \cdot 10^{12} + (abc_{10}) \cdot 10^9 + 789 \cdot 10^6 + 888 \cdot 10^3 + 000 \cdot 10^0$$
$$\equiv 20 - (abc_{10}) + 789 - 888 + 0 \ (mod\ 1001)$$
$$\equiv -79 - (abc_{10}) \text{ (mod 1001)}$$

Now, since we've stated that 16! is a multiple of 1001, we get

$$0 \equiv -79 - (abc_{10}) \text{ (mod 1001)} \longrightarrow (abc_{10}) \equiv -79 \equiv 922 \text{ (mod 1001)}$$

Obviously, abc_{10} is between 0 and 999 inclusive, so the only possible value is 922.

19. C Upon inspection, we can see that $f(x)$ has a root at $x = 1$. Synthetic dividing, this gives us

$$f(x) = (x - 1)(x^2 - 8x + 15) = (x - 1)(x - 3)(x - 5)$$

The key step is to take the entire expression modulo 3. Doing this, we get

$$f(x) = (x-1)(x-3)(x-5)$$
$$\equiv (x-1)(x)(x-2) \pmod 3$$
$$\equiv x(x-1)(x-2) \pmod 3$$

With this, it's easier to see that $|f(n)|$ is always a multiple of 3, so it's prime if and only if $f(n) = \pm 3$. For this to happen, two of the numbers $(x-1)$, $(x-3)$, and $(x-5)$ must be ± 1. This only occurs at $x = 2,4$. Plugging both values of x in, we get

$$f(2) = (1)(-1)(-3) = 3 \longrightarrow |f(2)| = 3$$
$$f(4) = (3)(1)(-1) = -3 \longrightarrow |f(4)| = 3$$

Both values we test work, so our final answer is $2 + 4 = 6$.

20. B The key is that all fourth powers modulo 16 are equivalent to either 0 or 1. As a result, a sum of six fourth powers must be 0-6 mod 16. If n is 7-15 mod 16, the equation can't have any solutions. Looking at the answer choices, we see that $3127 \equiv 7$ (mod 16).

For completion of the problem, note that (3,3,4,5,5,6) works for 2964, (1,3,3,3,5,7) works for 3270, and (0,2,4,5,6,6) works for 3489.

21. B The problem is equivalent to finding 41^{-1} (mod 181). This can be done in many ways−the following is one of them.

$$17 \equiv (-4) \cdot 41 \pmod{181}$$
$$7 \equiv (-2)(17) + 41 \equiv (-2)(-4) \cdot 41 + 41 \equiv 9 \cdot 41 \pmod{181}$$
$$3 \equiv 17 - 2(7) \equiv (-4) \cdot 41 - 2(9) \cdot 41 \equiv (-22) \cdot 41 \pmod{181}$$
$$1 \equiv 7 - 2(3) \equiv (9) \cdot 41 - 2(-22) \cdot 41 \equiv (53) \cdot 41 \pmod{181}$$

Thus, our answer is 53.

22. A Use the fact that $\binom{83}{58} = \binom{83}{25}$. The product can be written out as

$$\binom{83}{25} = \frac{(83)(82)\ldots(60)(59)}{(25)(24)\ldots(2)(1)}$$

Note that none of the numbers in the numerator or denominator are multiples of 29. Then, since 29 is prime, we can safely divide and find

$$\frac{(83)(82)\ldots(60)(59)}{(25)(24)\ldots(2)(1)} \equiv \frac{(25)(24)\ldots(2)(1)}{(25)(24)\ldots(2)(1)} \equiv 1 \pmod{29}$$

23. B For a number to have 6 positive factors, it must be of the form p^5 or p^2q for primes p, q. We only want the tenth smallest number, so we can just make a short list. We first list values of p^5:

$$2^5 = 32, 3^5 = 243, 5^5 = 3125, \ldots,$$

For values of p^2q, it's helpful to create cases based on the value of p^2

2^2: $2^2 \cdot 3 = 12, 2^2 \cdot 5 = 20, 2^2 \cdot 7 = 28, 2^2 \cdot 11 = 44, 2^2 \cdot 13 = 52,$
$2^2 \cdot 17 = 68, 2^2 \cdot 19 = 76$

3^2: $3^2 \cdot 2 = 18,\ 3^2 \cdot 5 = 45,\ 3^2 \cdot 7 = 63,\ 3^2 \cdot 11 = 99$

5^2: $5^2 \cdot 2 = 50, 5^2 \cdot 3 = 75$

7^2: $7^2 \cdot 2 = 98, 7^2 \cdot 3 = 147$

Looking at our list, we see that the smallest 10 numbers are 12, 18, 20, 28, 32, 44, 45, 50, 52, 63. Hence, our final answer is 63.

24. D We can factor n very nicely

$$= 2^{72} \cdot (9^{18}+1)(9^{18}-1)$$
$$= 2^{72} \cdot (9^{18}+1)(9^{9}+1)(9^{9}-1)$$
$$= 2^{72} \cdot (9^{18}+1)(9^{9}+1)(9^{6}+9^{3}+1)(9^{3}-1)$$

Both $(9^{18}+1)$ and $(9^{9}+1)$ are congruent to $1+1 \equiv 2 \pmod 4$, so they each have exactly one power of 2. (9^6+9^3+1) is equivalent to $1+1+1 \equiv 1 \pmod 2$, so it contains no powers of 2. As for (9^3-1), we can expand it out to find $9^3-1 = 728 = 2^3 \cdot 7 \cdot 13$; it has three powers of 2. As a result, we have in total $72+1+1+3 = 77$ powers of 2 in n.

25. C Since 37 is prime, Wilson's theorem gives that $36! \equiv -1 \equiv 36 \pmod{37}$. Then it's apparent that

$$33! = \frac{36!}{36 \cdot 35 \cdot 34} \equiv \frac{36}{36 \cdot 35 \cdot 34} \equiv 35^{-1} \cdot 34^{-1} \pmod{37}$$

Using the fact that $1 \equiv -36 \pmod{37}$ and $35 \equiv -2 \pmod{37}$, it's easy to see that $35^{-1} \equiv 18 \pmod{37}$. Similarly, using the fact that $34 \equiv -3 \pmod{37}$ we see that $34^{-1} \equiv 12 \pmod{37}$. Thus,

$$33! \equiv 35^{-1} \cdot 34^{-1} \equiv 18 \cdot 12 \equiv 216 \equiv 31 \pmod{37}$$

26. D We can write n in a form that's usable for Sophie Germain’s identity and factor the result

$$n = 8^6 + 9^8 = 81^4 + 4 \cdot 16^4$$
$$= (81^2 + 2 \cdot 16^2 + 2 \cdot 81 \cdot 16)(81^2 + 2 \cdot 16^2 - 2 \cdot 81 \cdot 16)$$
$$= (6561 + 512 + 2592)(6561 + 512 - 2592)$$
$$= (9665)(4481) = (5 \cdot 1933)(4481)$$

We're given that n is squarefree and has exactly 3 prime factors, so these must be our desired primes. Then our answer is $5 + 1933 + 4481 = 6419$.

27. B We use Euler's totient theorem to simplify the expression a bit. Since we know $\varphi(125) = \frac{4}{5} \cdot 125 = 100$, we only need to find $12^{76} \pmod{125}$. Now, we examine the binomial expansion

$$12^{76} = (2+10)^{76} = 2^{76} \cdot 10^0 \binom{76}{0} + 2^{75} \cdot 10^1 \binom{76}{1} + 2^{74} \cdot 10^2 \binom{76}{2} + \cdots$$

Every term after the three shown above contains a power of 10 greater than 10^2, so they are all 0 (mod 125); this makes them irrelevant to our sum. Then, all we must do is evaluate the three terms modulo 125. We can find the equivalence

$$12^{76} = 2^{76} + 2^{75} \cdot 10 \cdot 76 + 2^{74} \cdot 10^2 \cdot \frac{76 \cdot 75}{2} \pmod{125}$$
$$\equiv 2^{76} + 2^{76} \cdot 380 + 0 \pmod{125}$$
$$\equiv 2^{76} \cdot 381 \equiv 2^{76} \cdot 6 \equiv 2^{77} \cdot 3 \pmod{125}$$

by noting that the last term contains $10^2 \cdot 75$, so it is a multiple of 125. To find $2^{77} \pmod{125}$, we compute

$$2^7 = 128 \equiv 3 \pmod{125}$$
$$2^{35} \equiv 3^5 \equiv 243 \equiv -7 \pmod{125}$$
$$2^{70} \equiv (-7)^2 \equiv 49 \pmod{125}$$
$$2^{77} \equiv 2^{70} \cdot 2^7 \equiv 49 \cdot 3 = 147 \equiv 22 \pmod{125}$$

Hence, our final answer is $22 \cdot 3 = 66 \pmod{125}$.

28. B For notational convenience, let $n = \binom{44}{12}$. We want $n \pmod{10}$ so we take n modulo 2 and modulo 5. Finding $n \pmod{2}$ can be done by finding the number of twos inside of n. Using Legendre's formula, we get

$$\begin{aligned} 44!&:\ 22+11+5+2+1=41 \\ 12!&:\ 6+3+1=10 \\ 32!&:\ 16+8+4+2+1=31 \end{aligned}$$

So, there are $41-10-31=0$ twos in n, which tells us $n \equiv 1 \pmod{2}$. For n modulo 5, it's straightforward to use mass cancellation and Wilson's theorem, which gives us the fact that $4! \equiv -1 \pmod{5}$:

$$\begin{aligned} n &= \frac{44\cdot 43\cdot 42\cdot 41\cdot 40\cdot 39\cdot 38\cdot 37\cdot 36\cdot 35\cdot 34\cdot 33}{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} \\ &= \frac{(44\cdot 43\cdot 42\cdot 41)(39\cdot 38\cdot 37\cdot 36)(34\cdot 33)}{(12\cdot 11)(9\cdot 8\cdot 7\cdot 6)(4\cdot 3\cdot 2\cdot 1)}\cdot\frac{40}{10}\cdot\frac{35}{5} \\ &\equiv \frac{(-1)(-1)(2)}{(2)(-1)(-1)}\cdot 4\cdot 7 \equiv 3 \pmod{5} \end{aligned}$$

With Chinese Remainder Theorem, we can combine our two results together and find $n \equiv 3 \pmod{10}$, giving us our final answer.

29. B To remove the terminal zeroes, we need to divide out all the powers of 10 in 25!. There are way more 2s than 5s in 25!, so we look at the number of 5s; the calculation is standard

$$25!: 5+1=6$$

Now the problem is equivalent to finding the value of $\left(\frac{1}{10}\right)^6 \cdot 25! \pmod{10}$. We can split this up into modulo 2 and modulo 5. The quantity is clearly 0 (mod 2) because there are more 2s than 5s in 25!. It remains to find the quantity modulo 5, which we can do with some clever manipulation

$$\begin{aligned} &= \left(\frac{1}{10}\right)^6 \cdot 25! \\ &= \left(\frac{1}{10}\right)^6 5^5(5!)\underbrace{(1\cdot 2\cdot 3\cdot 4\cdot 6\cdot 7\ldots\cdot 23\cdot 24)}_{\text{25! without the multiples of five}} \\ &= \left(\frac{1}{10}\right)^6 5^5\cdot 5^1(1!)\quad\underbrace{(1\cdot 2\cdot 3\cdot 4)}_{\text{5! without the multiples of five}}\quad\underbrace{(1\cdot 2\cdot 3\cdot 4\cdot 6\cdot 7\ldots\cdot 23\cdot 24)}_{\text{25! without the multiples of five}} \\ &= \left(\frac{1}{2}\right)^6 (1!)\quad\underbrace{(1\cdot 2\cdot 3\cdot 4)}_{\text{5! without the multiples of five}}\quad\underbrace{(1\cdot 2\cdot 3\cdot 4\cdot 6\cdot 7\ldots\cdot 23\cdot 24)}_{\text{25! without the multiples of five}} \end{aligned}$$

The final expression has no factors of 5 within it, so we can multiply and divide modulo 5 with no worries. Wilson's theorem gives us $4! \equiv -1 \pmod{5}$, which can be used to quickly simplify our expression

$\equiv \left(\frac{1}{2}\right)^6 \pmod 5$
$\equiv 3^6 \equiv 4 \pmod 5$
where the last step uses the facts $2 \cdot 3 \equiv 1 \pmod 5$ and $3^4 = 81 \equiv 1 \pmod 5$. Thus, the last nonzero digit of 72! is $0 \pmod 2$ and $4 \pmod 5$, giving us a final answer of 4.

30. D Factoring, we see $2024 = 2^3 \cdot 11 \cdot 23$ has $(3+1)(1+1)(1+1) = 16$ positive divisors, and you double it because the denominator can also be negative. Thus, there are 32 integers n that work.