Answer Key:

1. True
2. $\frac{3}{4}$
3. -12
4. -2021
5. 7
6. 503
7. 16
8. 8
9. 47
10. $\frac{5}{3}$
11. 2
12. 6
13. 6
14. 42
15. 36
16. 258
17. $4 \cdot 4 + 4 + 4$
18. $4 \cdot (7 - \frac{7}{7})$
19. $5 \cdot 5 - \frac{11}{11}$
20. $\frac{11 \cdot 11 - 1}{5}$
21. J
22. Y
23. N
24. Y
25. 40

Solutions:

1. This is a true statement (AM-GM for 3 variables). For a simple proof, consider the following:
   We have
$$\begin{aligned}&a^3+b^3+c^3-3abc\\&=(a+b+c)(a^2+b^2+c^2-ab-bc-ac)\\&=\frac{1}{2}(a+b+c)((a-b)^2+(a-c)^2+(b-c)^2)\\&\geq 0\end{aligned}$$
   Since a, b, c are all positive and squares are all non-negative.
   There are other proofs for general n and for all real numbers, but this is a simple proof for positive and 3 variables.

2. We can express $\frac{1}{i^2-1}$ as $\frac{1}{2}\left(\frac{1}{i-1}-\frac{1}{i+1}\right)$. Thus, after the $i=4$ term, the series will start cancelling out, so our sum simplifies to $\frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}\right)=\frac{3}{4}$.

3. We can do some row operations to get that $\begin{vmatrix}0&1&2&3\\1&0&1&2\\2&1&0&1\\3&2&1&0\end{vmatrix}=\begin{vmatrix}0&1&2&3\\1&0&1&2\\3&1&1&3\\0&2&2&0\end{vmatrix}$. Taking minors for the last row, we get that this determinant is equal to $2\cdot\begin{vmatrix}0&2&3\\1&1&2\\3&1&3\end{vmatrix}-2\cdot\begin{vmatrix}0&1&3\\1&0&2\\3&1&3\end{vmatrix}=-12$.

4. For any root r_i, we have
$$r_i^{2022}+r_i^2+r_i+1=0.$$
   Since we know none of the roots are 0, we can divide the equation by r_i to get
$$r_i^{2021}+r_i+1+\frac{1}{r_i}=0.$$
   Thus, summing over all 2022 roots, we have $\sum_{i=1}^{2022} r_i^{2021}=-\sum_{i=1}^{2022} r_i-\sum_{i=1}^{2022} 1-\sum_{i=1}^{2022}\frac{1}{r_i}$
   We can find that the sum of the roots are 0, the sum of 2022 1's are 2022, and the sum of reciprocals of roots are $-\frac{1}{1}=-1$. Thus, we find that the sum is equal to -2021.

5. We can just find the units digit of each of the powers and sum them:

$$\sum_{i=1}^{10} i^i \bmod 10$$
$$= 1 + 4 + 7 + 6 + 5 + 6 + 3 + 6 + 9 + 0 \bmod 10$$
$$= 7.$$

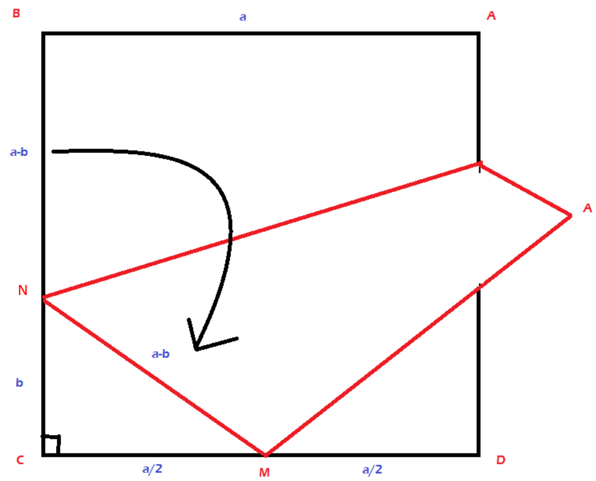
6. To find how many trailing zeros $2022!$ has, we need to find the number of 2's and $5's$ we have. Since a 10 consists of one 2 and one 5, we see that we simply need to count the number of 5's that appear in $2022!$ as we will have more 2's than 5's. Thus, we have $\left\lfloor\frac{2022}{5}\right\rfloor = 404, \left\lfloor\frac{404}{5}\right\rfloor = 80, \left\lfloor\frac{80}{5}\right\rfloor = 16, \left\lfloor\frac{16}{5}\right\rfloor = 3, \left\lfloor\frac{3}{5}\right\rfloor = 0$. Thus, there are a total of $404 + 80 + 16 + 3 = 503$ 5's, thus there are 503 trailing zeros.
The idea behind these divisions is that out of the 2022 numbers, only $\frac{1}{5}$ of them are going be a multiple of 5. Of those 404 multiples of 5, only $\frac{1}{5}$ of them are going to be a multiple of 25, and we repeat this until the power of 5 is greater than 2022.

7. We can count how many positive integers less than 30 are coprime with 30 by using Euler's totient function: $\phi(30) = 30\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right) = 8$. The idea behind this function is that $\frac{1}{2}$ of the positive integers less than 30 are multiples of 2, $\frac{1}{3}$ of them are multiples of 3 and $\frac{1}{5}$ of them are multiples of 5. We want the number of integers that are none of these, so we take the complement of the union of those sets.
Now, to find the number of integers coprime with 30 that is between 30 and 60, note that $GCD(x+30, 30) = GCD(x, 30)$, so there are also 8 integers coprime with 30 between 30 and 60. Thus, there are 16 integers total.

8. We can think of this problem as finding the number $mod\ 2, mod\ 3$, and $mod\ 5$, then using CRT to know there is a unique solution $mod\ 30$.
$$2^{3^5} mod\ 2 \equiv 0^{3^5} mod\ 2 = 0\ mod\ 2$$
$$2^{3^5} mod\ 3 \equiv (-1)^{3^5} mod\ 3 \equiv -1\ mod\ 3$$
$$2^{3^5} mod\ 5 \equiv 1^{\left\lfloor\frac{3^5}{4}\right\rfloor} \cdot 2^{3^5 mod\ 4} mod\ 5 \equiv 2^3 mod\ 5 \equiv 3\ mod\ 5$$
Thus, we need to find the positive integer less than 30 that satisfies these 3 constraints, and this is equal to 8.

9. Have fun counting! (there are 16 small, 9 big, 4 2s, 8 3s, 4 4s, 4 5s, 2 6s, so $16 + 9 + 4 + 8 + 4 + 4 + 2 = 47$).

10. Here is a picture of what the figure looks like:



Where $a = 1$, and we want to solve for $\frac{a-b}{b}$. From Pythagorean theorem, we can get the equation $(1-b)^2 = b^2 + \frac{1}{4}$. Solving for b, we get $b = \frac{3}{8}$. Thus, $a - b = \frac{5}{8}$, so $\frac{BN}{NC} = \frac{5}{3}$.

11. We claim that only 1 and 2 are not part of Pythagorean triples. For all odd integers x greater than or equal to 3, we can construct the triple $x, \frac{x^2-1}{2}, \frac{x^2+1}{2}$. We see this is a valid triple as $x^2 + \frac{x^4-2x^2+1}{4} = \frac{x^4+2x^2+1}{4}$. For all integers that are in the form of $2^k \geq 4$, note that we can simply multiply the triple $3-4-5$ by 2^{k-2} to get the triple with power of 2. Finally, for all even integers that are not powers of $2, x > 4$, then there exists an odd prime factor p. We can multiply the triple containing p by $\frac{x}{p}$ to get a valid triple containing x, so we are done.

12. Using Stewart's theorem, we have

$$man + dad = bmb + cnc$$
$$3 \cdot 4 \cdot 7 + 7d^2 = 6 \cdot 4 \cdot 6 + 8 \cdot 3 \cdot 8$$
$$7d^2 = 144 + 192 - 84$$
$$d = 6.$$

Alternatively, since this AD is an angle bisector, a useful simplification of Stewart's yields that $\overline{AD} = \sqrt{bc - mn} = \sqrt{6 \cdot 8 - 3 \cdot 4} = 6$.

13. We have that the probability of rolling a 6 is $\frac{1}{6}$.

Solution 1: Since we are rolling until we get one 6, that means that this follows a geometric probability distribution. The mean of a geometric probability distribution is $\frac{1}{p}$, so we have that the expected number of rolls is $\frac{1}{\frac{1}{6}} = 6$.

Solution 2: Alternatively, we can compute the expected value:

$$E = \sum i \cdot p(success)$$
$$= \sum i \cdot \left(\frac{5}{6}\right)^{i-1} \cdot \frac{1}{6}$$
$$= \frac{1}{6} \cdot \frac{\frac{5}{6}}{\left(1 - \frac{5}{6}\right)^2} \cdot \frac{6}{5} = 6$$

The last formula comes from $\sum i \cdot r^i = \frac{r}{(1-r)^2}$ for $-1 < r < 1$. This can be derived in the same way the formula for a geometric series is derived.

Solution 3: Intuitively, since we have a $\frac{1}{6}$ chance of rolling a 6, it'll take around 6 tries to get a 6 on average.

14. Solution 1: The mathy way is to set up "states" in which we move in-between whenever we roll a die. Let A be the initial state with no 6's rolled, and let B be the next state where we just rolled a 6. Then we have $A = \frac{5}{6}(A+1) + \frac{1}{6}(B+1)$ and $B = \frac{1}{6} \cdot 1 + \frac{5}{6}(A+1)$ and we want to solve for A, which gives $A = 42$.
Explanation: We have $\frac{5}{6}$ chance of not rolling a 6 and $\frac{1}{6}$ chance of rolling a 6 in state A, $e$ach incrementing the roll amount by 1. For state B, we have a $\frac{1}{6}$ chance of ending the game, and $\frac{5}{6}$ chance of restarting from A with an additional roll.

Solution 2: Alternatively, we can think about it as we expect to roll 6 times before we get our first 6, Thus, if we roll one more time, we have a $\frac{1}{6}$ chance of seeing a 6. Thus, for every 7 rolls, we expect a $\frac{1}{6}$ chance of rolling the double sixes, so the expected total is $7 \cdot 6$. This can be further generalized for more consecutive 6's, as you will see in question 16.
(credits to stack exchange)

15. We will do this in the states way as seen in the previous problem:
Let A be the initial state with no 6's rolled, and let B be the next state where we just rolled a 6. Then we have $A = \frac{5}{6}(A+1) + \frac{1}{6}(B+1)$ and $B = \frac{1}{6} \cdot 1 + \frac{4}{6}(A+1) + \frac{1}{6}(B+1)$. and we want to solve for A, which gives $A = 36$.
Explanation: We have $\frac{5}{6}$ chance of not rolling a 6 and $\frac{1}{6}$ chance of rolling a 6 in state A, $e$ach incrementing the roll amount by 1. For state B, we have a $\frac{1}{6}$ chance of ending the

game, $\frac{1}{6}$ chance of staying in the same state with an additional roll of 6, and $\frac{4}{6}$ chance of restarting from A with an additional roll.

16. We will expand on the second solution from problem 14. Since we expect it to take 42 rolls before we get a 66, then the following roll has a $\frac{1}{6}$ chance of being a 6. Thus, we expect it to take $(42+1)\cdot 6 = 258$ rolls to get three consecutive 6's.

17. The only solution is $4 \cdot 4 + 4 + 4$.

18. The only solution is $4 \cdot \left(7 - \frac{7}{7}\right)$.

19. The only solutions is $5 \cdot 5 - \frac{11}{11}$.

20. The only solution is $\frac{11 \cdot 11 - 1}{5}$.

21. This sequence follows JANUARY, FEBUARY, MARCH, APRIL, MAY, JUNE, so the answer would be J.

22. This sequence follows the top row of a QWERTY keyboard, so the answer would be Y.

23. This sequence follows HYDROGEN, HELIUM, LITHIUM, BERYLLIUM, BORON, CARBON, so the answer would be N.

24. This sequence follows $A = 1, D = 4, I = 9, P = 16$, so the answer would be Y.

25. We have

$$\begin{aligned}
&0 + 2 + 6 + 4 + 2 \\
&+0 + 0 + 0 + 0 + 1 \\
&+0 + 0 + 0 + 0 + 0 \\
&+0 + 2 + 0 + 4 + 5 \\
&+0 + 0 + 0 + 0 + 1 \\
&+13 \\
&= 40
\end{aligned}$$