MAΘ National Convention 2023 Theta Bowl Solutions

0. 11. $35x^2 - 11x - 6$ factors into $(7x+2)(5x-3)$; $A = 7, B = 2,\ C = 5; D = -3; A + B + C + D = 7 + 2 + 5 - 3 = 11$.

1. 26. $A = \log_7 16$. Using logarithm rules, $A = \frac{(\log 10 + \log 2 + \log 5)\log 4}{\log 7} = \frac{(\log 10 + \log 10)\log 4}{\log 7} = \frac{(1+1)\log 4}{\log 7} = \frac{2\log 4}{\log 7} = 2\log_7 4 = \log_7 16$. $B = 7$. Using logarithm and exponent rules, $B = e^{\ln(\frac{e^{\ln 28 - \ln 2}}{e^{\ln 32 - 4\ln 2}})} = e^{\ln(\frac{e^{\ln 28 - \ln 2}}{e^{\ln 32 - \ln 16}})} = e^{\ln(\frac{e^{\ln 14}}{e^{\ln 2}})} = e^{\ln(\frac{14}{2})} = e^{\ln 7} = 7$. $f(x) = \frac{10(x^2-4)(x^3-125)}{(x^2+5x+6)(x^3-2x^2-25x+50)(x+7)}$, which can be further factored into $\frac{10(x+2)(x-2)(x-5)(x^2+5x+25)}{(x+2)(x+3)(x+5)(x-5)(x-2)}$. Cancelling out the common factors in the numerator and denominator leaves $\frac{10(x^2+5x+25)}{(x+3)(x+5)}$. $C = 5$. C the sum of the numbers that make the cancelled factors equal zero. The cancelled factors are $(x+2)$, $(x-2)$, and $(x-5)$, so the holes are at $x = -2, 2, 5$. The sum of these is 5. $D = 2$. The vertical asymptotes of the graph occur at $x = -3$ and $x = -5$, while the horizontal asymptote occurs at $y = 10$ (the ratio of the leading coefficients since the numerator and denominator have the same degree. $D = -3 - 5 + 10 = 2$. $B^A + CD = 7^{\log_7 16} + 5 \times 2 = 16 + 10 = 26$.

2. $\frac{720}{11}$. $A = 27\sqrt{3}$. The area of a regular hexagon with side length s is $\frac{3s^2\sqrt{3}}{2}$. So $A = \frac{3(3\sqrt{2})^2\sqrt{3}}{2} = 27\sqrt{3}$. $B = \frac{33\sqrt{3}}{4}$. Drawing a line segment from the end of the shorter base to the longer base so that it is perpendicular to both bases creates a right triangle. The length of this segment can be h, and the triangle contains the trapezoid's base angle of 60° across from the leg h. The shorter leg is $\frac{7-4}{2} = \frac{3}{2}$, so h must be $\frac{3\sqrt{3}}{2}$ because this triangle is a 30°-60°-90° triangle. h is also the height of the trapezoid, so the area is $\frac{(b_1+b_2)h}{2} = \frac{(4+7)\frac{3\sqrt{3}}{2}}{2} = \frac{33\sqrt{3}}{4}$. $C = \frac{20}{1+\sqrt{2}}$. The side length of the octagon can be x. The isosceles triangles cut out of the square have hypotenuses of x, which means their legs would each be $\frac{x\sqrt{2}}{2}$. The length of one side of the square is $80 \div 4 = 20$, which is also equal to $\frac{x\sqrt{2}}{2} + x + \frac{x\sqrt{2}}{2}$. These can be set equal to solve for x: $\frac{x\sqrt{2}}{2} + x + \frac{x\sqrt{2}}{2} = x + x\sqrt{2} = x(1+\sqrt{2}) = 20$; $x = \frac{20}{1+\sqrt{2}}$. $D = 1 + \sqrt{2}$. The space diagonal of a cube of side length x is $x\sqrt{3}$ (this can be found via Pythagorean theorem). Thus, setting the space diagonal $\sqrt{3} + \sqrt{6}$ equal to $x\sqrt{3}$ will give x, the side length of the cube. $\sqrt{3} + \sqrt{6} = x\sqrt{3}$; $x = \frac{\sqrt{3}+\sqrt{6}}{\sqrt{3}} = 1 + \sqrt{2}$. $\frac{ACD}{B} = \frac{(27\sqrt{3})(\frac{20}{1+\sqrt{2}})(1+\sqrt{2})}{(\frac{33\sqrt{3}}{4})} = \frac{27\times 20}{\frac{33}{4}} = \frac{27\times 20\times 4}{33} = \frac{9\times 20\times 4}{11} = \frac{720}{11}$.

3. $-480 + 288\pi$. $A = \frac{8}{3}$. $4x^2 - 9y^2 - 16x - 54y - 101 = 0$ can be rewritten as $\frac{(x-2)^2}{9} - \frac{(y+3)^2}{4} = 1$. The latus rectum is $\frac{2b^2}{a} = \frac{2(2)^2}{3} = \frac{8}{3}$. $B = -6$. The slope of the line that is perpendicular to $2x + 5y = 3$ is the negative reciprocal of the slope of this line. The slope of this line is $\frac{-A}{B} = \frac{-2}{5}$ so the slope of the perpendicular line is $\frac{5}{2}$. The equation of the perpendicular line can be written as $(y+8) = \frac{5}{2}(x+5)$ and expanded to $5x - 2y = -9$. $a + b + c = 5 - 2 - 9 = -6$. $C = 30$. The points that could complete the parallelogram are (12, 2), (0, 4), and (8, 4). The sum of these coordinates is $12 + 2 + 0 + 4 + 8 + 4 = 30$. $D = 288\pi$. The equation of the ellipse can be rewritten as $\frac{(x-6)^2}{9} + \frac{(y-5)^2}{4} = 1$. The equation of the lines tangent and perpendicular to its major and minor axes are $y = 7$, $y = 3$, $x = 3$, and $x = 9$. Rotating the rectangle formed by these lines around the y-axis creates a cylinder with a cylinder shaped hole in the middle with volume $\pi \times 4 \times (9^2 - 3^2) = 288\pi$. $ABC + D = \left(\frac{8}{3}\right)(-6)(30) + 288\pi = -480 + 288\pi$.

4. $3218 + \sqrt{6}$. $A = 2648$. $B = 495$. Set $x = 5.3\overline{49}$. $10x = 53.\overline{49}$ and $1000x = 5349.\overline{49}$. Subtracting the first equation from the second gets rid of the repeating part: $1000x - 10x = 5349.\overline{49} - 53.\overline{49}$; $990x = 5296$;

$x = \frac{5296}{990} = \frac{2648}{495}$. $C = -3$. For $n+2$, n, and $2n-1$ to form a geometric sequence, $\frac{n}{n+2}$ must equal $\frac{2n-1}{n}$. Setting these equal and cross multiplying yields $2n^2 + 3n - 2 = n^2$. Bringing all terms to one side gives $n^2 + 3n - 2 = 0$, and n can be found via the quadratic formula: $n = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-3 \pm \sqrt{3^2-4(1)(-2)}}{2(1)} = \frac{-3 \pm \sqrt{17}}{2}$. The sum of the possible values of n is $\frac{-3+\sqrt{17}}{2} + \left(\frac{-3-\sqrt{17}}{2}\right) = \frac{-3}{2} + \left(\frac{-3}{2}\right) = -3$. $D = 6 + \sqrt{6}$. $a = 5 + \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}}$, so $a - 5 = \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}}$. Substituting in $a-5$ gives $a - 5 = \sqrt{a + (a-5)}$. Squaring both sides of the equation gives $(a-5)^2 = a + a - 5$, which can be expanded and a can be solved for: $a^2 - 10a + 25 = 2a - 5$; $a^2 - 12a + 30 = 0$. To solve for a, complete the square: $a^2 - 12a + 36 = -30 + 36$; $(a-6)^2 = 6$; $a - 6 = \pm\sqrt{6}$; $a = 6 \pm \sqrt{6}$. However, $6 - \sqrt{6}$ is extraneous because square roots are never negative. $E = 72$. $\frac{(n+2)!}{7(n-1)!}$ simplifies to $\frac{(n+2)(n+1)(n)}{7}$; when $n = 7$, this equals 72. $A + B + C + D + E = 2648 + 495 + (-3) + 6 + \sqrt{6} + 72 = 3218 + \sqrt{6}$.

5. $\frac{1080}{11}$. $A = 3$. The set of points equidistant from $(3, 4)$ and $(-1, 2)$ is the perpendicular bisector of the line segment with endpoints $(3, 4)$ and $(-1, 2)$. The line must pass through the midpoint of this line segment and the slope must be perpendicular to the slope of the line segment. The midpoint of the line segment is $(\frac{3+(-1)}{2}, \frac{4+2}{2}) = (\frac{2}{2}, \frac{6}{2}) = (1, 3)$. The slope of the line segment is $\frac{4-2}{3-(-1)} = \frac{2}{4} = \frac{1}{2}$. The slope of the line must be the negative reciprocal of this: -2. The equation of the line can be expressed as $(y-3) = -2(x-1)$ and the y-intercept can be found by plugging 0 in for x: $(y-3) = -2(0-1) = -2(-1) = 2$; $y = 2 + 3 = 5$. The sum of the slope and y-intercept of the line is $-2 + 5 = 3$. $B = -3$. These lines are parallel if they have the same slope, meaning $\frac{(w-2)-5}{1-3} = \frac{(5-w)-13}{2-3}$ which simplifies into $\frac{w-7}{-2} = \frac{-8-w}{-1}$. Cross multiplying yields $7 - w = 16 + 2w$. Solving for w gives $3w = -9$; $w = -3$. $C = 5$. $(f \circ g)(1) = f(g(1)) = f(1) = 0$. $g((f \circ g)((f \circ g^{-1})(1))) = g((f \circ g)(f(1))) = g((f \circ g)(0)) = g((f(3)) = g(4) = -5$. $(f \circ g)(1) \times g((f \circ g)((f \circ g^{-1})(1))) = 0 - (-5) = 5$. $D = -6$. $(g \circ f)(3) = g(f(3)) = g(4) = -5$. $g(3) = -4$. $g^{-1}(-4) = 3$. $(g \circ f)(3) + g(3) + f^{-1}(-4) = -5 - 4 + 3 = -6$. $E = \frac{4}{11}$. $h(7) = h(3x+1)$: solving for x gives $7 = 3x + 1$; $3x = 6$; $x = 2$. $h(3(2)+1) = \frac{3(2)-2}{2+9} = \frac{4}{11}$. $F = 1$. $h\left(\frac{35}{2}\right) = h(3x+1)$: solving for x gives $\frac{35}{2} = 3x + 1$; $3x = \frac{33}{2}$; $x = \frac{11}{2}$. $h\left(3\left(\frac{11}{2}\right) + 1\right) = \frac{3\left(\frac{11}{2}\right)-2}{\frac{11}{2}+9} = 1$. $ABCDEF = (3)(-3)(5)(-6)(\frac{4}{11})(1) = \frac{1080}{11}$.

6. 4216. $A = 5$. The centroid of the triangle defined by $(2, 3)$, $(-4, 2)$, $(7, 5)$ is $(\frac{2-4+7}{3}, \frac{3+2+5}{3}) = (\frac{5}{3}, \frac{10}{3})$. $x + y = \frac{5}{3} + \frac{10}{3} = \frac{15}{3} = 5$. $B = \frac{33}{7}$. Calling the length of XW a and using angle bisector theorem gives $\frac{9-x}{10} = \frac{x}{11}$. Cross multiplying gives $11(9-x) = 10x$; $99 - 11x = 10x$; $99 = 21x$; $x = \frac{99}{21} = \frac{33}{7}$. $C = 1250$. One side of the enclosure can be x and the other can be y. The length of fencing is $100 = 2x + y$. The area of the enclosure is xy. $y = 100 - 2x$, which can be plugged into the formula for the area to give $x(100-2x) = 100 - 2x^2$. The area (this quadratic) will be maximized when $x = \frac{-b}{2a} = \frac{-100}{2(-2)} = 25$. $y = 100 - 2(25) = 50$. $xy = 25 \times 50 = 1250$. $D = 26$. The sum of the interior angles (in degrees) of a regular decagon is $180(n-2) = 180(10-2) = 180(8) = 1440$. The measure of an interior angle of a regular decagon is $\frac{1440}{10} = 144$. The number of diagonals in an icosagon is $\frac{n(n-3)}{2} = \frac{20(20-3)}{2} = \frac{20(17)}{2} = 170$. The positive difference between these two is $170 - 144 = 26$. $\frac{BC}{AD} = \frac{(\frac{33}{7})(1250)}{(5)(26)} = \frac{4125}{91}$. The sum of the numerator and denominator is $4125 + 91 = 4216$.

7. $-24213 - 24188i$. $A = 75i$. $(10+5i)(3+6i) = 30 + 75i + 30i^2 = 30 + 75i - 30 = 75i$. $B = \frac{4-3i}{3}$. $\frac{10+5i}{3+6i} = \frac{(10+5i)(3-6i)}{(3+6i)(3-6i)} = \frac{30-45i-30i^2}{9-36i^2} = \frac{30-45i+30}{9+36} = \frac{60-45i}{45} = \frac{4-3i}{3}$. $C = 24$. $|3-4i| = \sqrt{3^2+4^2} = 5$; $x < 5$, so the product of all possible positive integer values of x is $1 \times 2 \times 3 \times 4 = 24$. $D = -1012i - 1012$. The powers of i cycle every four: $i + 2i^2 + 3i^3 + 4i^4 + \cdots + 400i^{400} = i - 2 - 3i + 4 + \cdots + 400 =$

$(-2i+2)(100) = -200i + 200$. $AB + CD = (75i)\left(\frac{4-3i}{3}\right) + (24)(-200i+200) = 25i(4-3i) + (-4800i + 4800) = 100i - 75i^2 - 4800i + 4800 = 4875 - 4700i$.

8. $\frac{-375\pi\sqrt{7}}{56}$. $A = 25\pi$. $x^2 + y^2 - 8x - 2y - 8 = 0$ factors into $(x-4)^2 + (y-1)^2 = 25$. This is a circle with radius 5, so the area is 25π. $B = \frac{225}{56}$. This region is a right triangle, and the lengths of its legs can be found by finding the distance the origin is from the x- and y-intercepts of the line $4x - 7y + 15 = 0$. Plugging in 0 for x gives the y-intercept: $4(0) - 7y + 15 = 0$; $y = \frac{15}{7}$. Plugging in 0 for y gives the x-intercept: $4x - 7(0) + 15 = 0$; $x = \frac{-15}{4}$. The area of this triangle is $\frac{1}{2}bh$ where b can be the distance from the x-intercept to the origin, and h can be the distance from the y-intercept to the origin: $\frac{1}{2}bh = \frac{1}{2}\left(\frac{15}{4}\right)\left(\frac{15}{7}\right) = \frac{225}{56}$. $C = \frac{\sqrt{7}}{4}$. $16x^2 + 9y^2 - 32x + 18y - 119 = 0$ can be expressed as $\frac{(x-1)^2}{9} + \frac{(y+1)^2}{16} = 1$. The eccentricity of a conic is $\frac{c}{a}$. $a^2 = 16$; $a = 4$. $a^2 - b^2 = c^2$; $c = \sqrt{a^2 - b^2} = \sqrt{16-9} = \sqrt{7}$. $\frac{c}{a} = \frac{\sqrt{7}}{4}$. $D = \frac{-15}{4}$. $x = y^2 - 4$ can be rewritten as $x + 4 = y^2$. The vertex of this parabola is $(-4, 0)$. $4p = 1$; $p = \frac{1}{4}$. The focus is thus at $(-4 + \frac{1}{4}$, $0)$, so $x + y = -4 + \frac{1}{4} + 0 = \frac{-15}{4}$. $\frac{ABC}{D} = \frac{(25\pi)(\frac{225}{56})(\frac{\sqrt{7}}{4})}{\frac{-15}{4}} = (25\pi)\left(\frac{225}{56}\right)\left(\frac{\sqrt{7}}{4}\right)\left(\frac{-4}{15}\right) = \frac{-375\pi\sqrt{7}}{56}$.

9. 9605. $A = \frac{16}{3}$. $\sum_{n=-2}^{\infty}\left(\frac{3}{4}\right)^{n+1} = \left(\frac{3}{4}\right)^{-1} + \left(\frac{3}{4}\right)^{0} + \left(\frac{3}{4}\right)^{1} + \left(\frac{3}{4}\right)^{2} + \cdots = \frac{a}{1-r} = \frac{\left(\frac{3}{4}\right)^{-1}}{1-\frac{3}{4}} = \frac{\frac{4}{3}}{\frac{1}{4}} = \frac{16}{3}$. $B = 25$. $\prod_{n=0}^{\infty}(5)^{\left(\frac{1}{2}\right)^n} = 5^1 + 5^{\frac{1}{2}} + 5^{\frac{1}{4}} + 5^{\frac{1}{8}} + \cdots = 5^{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots} = 5^{\frac{a}{1-r}} = 5^{\frac{1}{1-\frac{1}{2}}} = 5^2 = 25$. $C = \frac{3}{4}$. $\frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81} + \cdots$ can be expressed as a sum of infinite geometric series: $\left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots\right) + \left(\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots\right) + \left(\frac{1}{27} + \frac{1}{81} + \cdots\right) + \left(\frac{1}{81} + \cdots\right) + \cdots = \frac{\frac{1}{3}}{1-\frac{1}{3}} + \frac{\frac{1}{9}}{1-\frac{1}{3}} + \frac{\frac{1}{27}}{1-\frac{1}{3}} + \frac{\frac{1}{81}}{1-\frac{1}{3}} + \cdots = \frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \frac{1}{54} + \cdots = \frac{\frac{1}{2}}{1-\frac{1}{3}} = \frac{3}{4}$. $E = \sum_{n=3}^{\infty}\frac{2}{n^2-2n} = \sum_{n=3}^{\infty}\left(\frac{1}{n-2} - \frac{1}{n}\right)$ via partial fraction decomposition. $\sum_{n=3}^{\infty}\left(\frac{1}{n-2} - \frac{1}{n}\right) = \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \cdots$. All the terms cancel out except $\frac{1}{1} + \frac{1}{2} = \frac{3}{2}$. $D = 9455$. The sum of the squares of the first 30 positive integers is $\frac{(n)(n+1)(2n+1)}{6} = \frac{(30)(31)(61)}{6} = 9455$. $ABCE + D = \frac{16}{3} \cdot 25 \cdot \frac{3}{4} \cdot \frac{3}{2} + 9455 = 150 + 9455 = 9605$.

10. 7560. $A = \frac{1}{3}$. Regardless the of the first two rolls, the last roll has a $\frac{1}{3}$ chance of making the sum a multiple of 3. $B = \frac{4}{9}$. The probability that a number selected is negative is $\frac{1}{3}$. For the product to be negative, exactly one of the two numbers must be negative: $\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = 2\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{4}{9}$. $C = 10080$. The number of distinct combinations with O as the first letter is 7!, the number of distinct combinations with U as the first letter is $\frac{7!}{2!}$, and the number of distinct combinations with I as the first letter is $\frac{7!}{2!}$. The sum of these is $7! + \frac{7!}{2!} + \frac{7!}{2!} = 10080$. $\frac{AC}{B} = \frac{\left(\frac{1}{3}\right)(10080)}{\frac{4}{9}} = 7560$.

11. $904 + 16\pi\sqrt{330}$. $A = 40$. The area of this segment is $\frac{120}{360}\pi\left(\frac{10}{2}\right)^2 - \frac{1}{2}\left(\frac{10}{2}\right)\left(\frac{10}{2}\right)\sin 120° = \frac{25\pi}{3} - \frac{25\sqrt{3}}{4} = \frac{100\pi - 75\sqrt{3}}{12}$. $A = 100 - 75 + 3 + 12 = 40$. $B = 3\sqrt{55}\pi$. The slant height of the cone is the radius: 8. The circumference of the base of the cone is $\frac{135}{360}(2\pi \times 8) = 6\pi$. The radius of the base can be found by solving $2\pi r = 6\pi$; $r = 3$. The height of the cone $h = \sqrt{8^2 - 3^2} = \sqrt{55}$. The volume of this cone is $\frac{\pi r^2 h}{3} = \frac{\pi(3)^2(\sqrt{55})}{3} = 3\sqrt{55}\pi$. $C = \frac{113}{5}$. Knowing that medians split each other in 2:1 ratios, the median from Y can be given a length of $3a$ and the median from Z can be given a length $3b$. Pythagorean theorem gives $(2a)^2 + b^2 = \left(\frac{7}{2}\right)^2$ and $a^2 + (2b)^2 = \left(\frac{8}{2}\right)^2$. Adding these together, dividing both sides by 5 and multiplying

both sides by 4 gives $4a^2 + 4a^2 = \frac{113}{5}$. This is equal to $YZ^2 = (2a)^2 + (2b)^2 = \frac{113}{5}$. $D = \frac{16\sqrt{2}}{3}$. The volume of a tetrahedron with side s is $\frac{s^3}{6\sqrt{2}}$. The volume of a tetrahedron with side length 4 is $\frac{4^3}{6\sqrt{2}} = \frac{32}{3\sqrt{2}} = \frac{16\sqrt{2}}{3}$. $E = \sqrt{3}$. The surface area of a tetrahedron with side length 4 is $4 \times \frac{1^2\sqrt{3}}{4} = \sqrt{3}$. $AC + BDE = (40)\left(\frac{113}{5}\right) + \left(3\sqrt{55}\pi\right)\left(\frac{16\sqrt{2}}{3}\right)\left(\sqrt{3}\right) = 904 + 16\pi\sqrt{330}$.

12. $\frac{-729+\sqrt{13}}{2}$. $A = -364$. $\left|\left||x-6|-6\right|-6\right| = 8$. $\left||x-6|-6\right| - 6 = 8$ or $\left||x-6|-6\right| - 6 = -8$: $\left||x-6|-6\right| = 14$; $\left||x-6|-6\right| = -2$ is the other possibility but an absolute value cannot be negative. $|x-6| - 6 = 14$ or $|x-6| - 6 = -14$. $|x-6| = 20$; $|x-6| = -8$ is the other possibility but an absolute value cannot be negative. $x - 6 = 20$ or $x - 6 = -20$. $x = 26, -14$. The product of these two numbers is $26(-14) = -364$. $B = 0$. $\frac{x-2}{2x+2} > 2$ can be rewritten as $\frac{x-2}{2x+2} - \frac{4x+4}{2x+2} > 0$. $\frac{x-2-4x-4}{2x+2} > 0$. $\frac{-3x-6}{2x+2} > 0$. The critical points are $x = -1$ (makes the denominator zero) and $x = -2$ (makes both sides equal). Checking intervals between critical points shows that $-2 < x < -1$. The number of integers that lie in this interval is 0. $C = \frac{-1+\sqrt{13}}{2}$. $x^2 + 2nx - n + 3$ has exactly one real root if its determinant is zero: $b^2 - 4ac = 0 = (2n)^2 - 4(1)(-n+3) = 4n^2 + 4n - 12$. Using the quadratic formula: $n = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-(4)\pm\sqrt{(4)^2-4(4)(-12)}}{2(4)} = \frac{-4\pm\sqrt{208}}{8} = \frac{-1\pm\sqrt{13}}{2}$. The larger value of n is $\frac{-1+\sqrt{13}}{2}$. $D = 0$. $(x^2 - 4x + 4)^{x^2+4x+3} = 1$. $x^2 - 4x + 4$ must either equal 1, $x^2 + 4x + 3$ must equal 0, or $x^2 - 4x + 4$ must equal -1 and $x^2 + 4x + 3$ is even. $x^2 - 4x + 4 = 1$ means $x^2 - 4x + 3 = 0 = (x-3)(x-1)$; $x = 3, 1$. $x^2 + 4x + 3 = 0 = (x+3)(x+1)$; $x = -3, -1$. $x^2 - 4x + 4 = -1$; $(x-2)^2 = -1$, which is impossible. $3 + 1 - 3 - 1 = 0$. $A + B + C + D = -364 + 0 + \frac{-1+\sqrt{13}}{2} + 0 = \frac{-729+\sqrt{13}}{2}$.

13. 144000. $A = \frac{1400}{3}$.

Concentration of solution	Amount of solution	Total amount of hydrochloric acid
0.05	A	$0.05A$
0.02	$700 - A$	$0.02(700 - A)$
0.04	700	$0.04(700) = 28$

Creating and using the above table, $0.05A + 0.02(700 - A) = 28$; $0.05A + 14 - 0.02A = 28$; $0.03A + 14 = 28$; $0.03A = 14$; $A = \frac{14}{0.03} = \frac{1400}{3}$. $B = \frac{24}{7}$. Mr. Fayiga's rate is 1 house per 6 days $(\frac{1}{6})$, and Mr. Lu's rate is 1 house per 8 days $(\frac{1}{8})$. Their combined rate is $\frac{1}{6} + \frac{1}{8} = \frac{7}{24}$. It takes them d days to paint 1 house together: $\frac{7}{24}d = 1$. $d = \frac{24}{7}$. $C = 90$. The rate it takes for the bathtub to fill is 1 bathtub per 30 minutes $(\frac{1}{30})$, and the rate it takes for the bathtub to empty is 1 bathtub per 45 minutes $(\frac{-1}{45})$. With the drain open, the bathtub's fill rate will be $\frac{1}{30} - \frac{1}{45} = \frac{1}{90}$. It takes the bathtub m minutes to fill/overflow with the drain open: $\frac{1}{90}m = 1$; $m = 90$. $ABC = \left(\frac{1400}{3}\right)\left(\frac{24}{7}\right)(90) = 144000$.

14. 15961. $A = 439$. $\begin{vmatrix} 1 & 12 & -2 \\ -3 & 5 & 3 \\ 5 & 15 & 4 \end{vmatrix} = 1\begin{vmatrix} 5 & 3 \\ 15 & 4 \end{vmatrix} - 12\begin{vmatrix} -3 & 3 \\ 5 & 4 \end{vmatrix} - 2\begin{vmatrix} -3 & 5 \\ 5 & 15 \end{vmatrix} = 1(20-45) - 12(-12-15) - 2(-45-25) = 1(-25) - 12(-27) - 2(-70) = -25 + 324 + 140 = 439$. $B = 42$. $\begin{bmatrix} 5 & 6 \\ -2 & 3 \end{bmatrix}^2 = \begin{bmatrix} 5 & 6 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} (5)(5)+(6)(-2) & (5)(6)+(6)(3) \\ (-2)(5)+(3)(-2) & (-2)(6)+(3)(3) \end{bmatrix} = \begin{bmatrix} 13 & 48 \\ -16 & -3 \end{bmatrix}$. The sum of the elements of this matrix is $13 + 48 - 16 - 3 = 42$. $C = -56$. The determinant of the transpose of a given matrix is the same as the determinant of that matrix. $\begin{vmatrix} 4 & 16 \\ 2 & -6 \end{vmatrix} = (4)(-6) - (16)(2) = -24 - 32 = -56$.

$D = 31.$ $\begin{bmatrix} 3 & 4 \\ -9 & 5 \\ 7 & -2 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} (3)(-1)+(4)(3) & (3)(-3)+(4)(2) \\ (-9)(-1)+(5)(3) & (-9)(-3)+(5)(2) \\ (7)(-1)+(-2)(3) & (7)(-3)+(-2)(2) \end{bmatrix} = \begin{bmatrix} 9 & -1 \\ 24 & 37 \\ -13 & -25 \end{bmatrix}$. The sum of the elements of this matrix is $9 - 1 + 24 + 37 - 13 - 25 = 31$. $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} 439 & 42 \\ -56 & 31 \end{vmatrix} = (439)(31) - (42)(-56) = 15961.$