Theta School Bowl Answers
2024 National Convention

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1. 97
2. 178
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4. $160 + 76\sqrt{2}$
5. $\frac{26}{3}$
6. -4
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10. -14
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12. 11
13. $12\sqrt{3}$
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0. **22**

A: The positive values of k such that $100 \leq k^2 \leq 1000$ are $10 \leq k \leq 31$. Therefore, there are 31 -10 + 1 = <u>22</u> perfect squares.
B: 100 + (n-1) 2 = 1000 → n= <u>451</u>
C: 102 + (n-1)6 = 996 → n= <u>150</u>
D: From Part A, there are 451 integers in S that are divisible by 2. Using the same method, there are 300 integers in S that are divisible by 3. There are 150 integers in S that are divisible by 2 and 3. By Principle of Inclusion and Exclusion, 451 + 300 -150 = <u>601</u>

22 + 451 + 150 – 601 = **22**

1. **97**

A: Recall that (rate)(distance)=time. Let *k be* Kevin's rate of swimming and let *c* be the rate of the river's current. We have the following equations (10)(r-c)=20 and (5)(r+c) =20. Therefore, r=3 and c=1. A=1
B: Let point B be (1,3) and point R be (5,9). In order to find the shortest path from B to R where Angela touches the line x=-1, we should reflect R over the line x=-1 to some point R' and solve for the distance BR'. R' = (-7,9). Therefore, B = 10.
C: 1 printer can print 100 sheets in 10 minutes or 600 seconds → 1 printer can print one sheet in 6 seconds. C=6
D. Let *n, d, w* be the number days that it takes Nathan, Daniel, and Will to build a house respectively. We have the following equations.

$$\frac{1}{n}+\frac{1}{w}=\frac{7}{24}$$
$$\frac{1}{w}+\frac{1}{d}=\frac{5}{24}$$
$$\frac{1}{n}+\frac{1}{d}=\frac{1}{4}$$

Sum the three equations and divide by 2 to see how many houses Nathan, Daniel, and William working together can build in a day.

$$\frac{1}{n}+\frac{1}{d}+\frac{1}{w}=\frac{3}{8}.$$

Thus, it takes them 8/3 days to build a house. To build 30 houses, the amount of time it would take is, D=30*8/3=80

2. **178**

A: The remainder must be one less degree than $x^2 - 5x + 6$. Therefore, it must be linear. We can call the remainder Ax + B. For some quotient q(x), we have that $x^4 + 7 = (x^2 - 5x + 6)\big(q(x)\big) + Ax + B$. Plugging in the roots (2, 3) we get a system of two linear equations for A and B.

$$2A + B = 23$$
$$3A + B = 88$$

Therefore, A = 65 and B = -107
65*4-107 = 153
B: $x^4 - 8x^2 + 7 = (x^2 - 1)(x^2 - 7)$. Therefore, the solutions are ± 1 and $\pm 2\sqrt{2}$. The sum of the squares of all solutions is 1 + 1 + 7 + 7 = 16
C: For some x, we know that $2x - 2$ = k and $x^2 + 7 = 1943$. From the second equation, we know that $x = \pm 44$. Therefore, k is either $2(44) - 2 = 86$ *or* $2(-44) - 2 = -90$. Sum these to find C=-4
D: Use Simon's Favorite factoring trick to factor the equation as:

$$(m - 4)(n - 1) = 7$$

Because both m and n are positive integers and 7 is prime, the only two pairs of m and n are (11, 2) and (5, 8). Both sum to 13.

153+16+(-4)+13 =**178**

3. **44**

A: The information given by the problem gives us the following three equations to solve for a, b, and c.

$$c = -1$$
$$9a + 3b - 1 = 2$$
$$36a + 6b - 1 = 41$$

Solve to find (a, b, c) = (2, -5, -1). We want to find $q(2) = 2(4) - 5(2) - 1 = -3$
B: Let the three roots be a, $a+d$, and $a-d$. By Vieta's formulas, we know

$$a + a + d + a - d = \frac{9}{2}$$
$$3a = \frac{9}{2} \rightarrow a = 3/2$$

Because a is a root of the equation, f(a) = 0. Set f(3/2) equal to zero to solve for k.

$$0 = 2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 + k\left(\frac{3}{2}\right) + 12$$
$$k = 1$$

$$\sum_{i=0}^{5} g(i) = \sum_{i=0}^{5} i^2 - \sum_{i=0}^{5} i + \sum_{i=0}^{5} 1$$

Use the formulas for sum of squares ($\frac{i(i+1)(2i+1)}{6}$) and sum of consecutive integers ($\frac{i(i+1)}{2}$).

$$5 * 6 * \frac{11}{6} - 5 * \frac{6}{2} + 6 = 46$$

C =46
A + B = -2
A + B+ C =44

4. $\mathbf{160 + 76\sqrt{2}}$

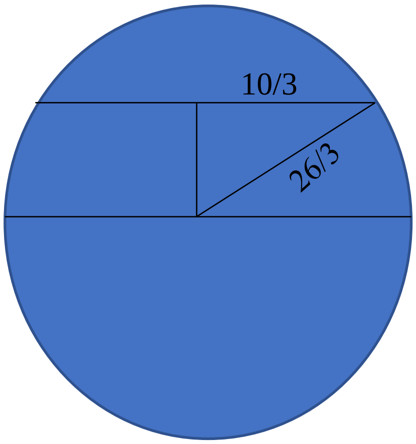
A: The area of the octagon in terms of its side length is given by the formula $2s^2(1+\sqrt{2})$ = $8 + 8\sqrt{2}$. It can also be calculated synthetically through 45-45-90 triangles.
B: We will consider the four isosceles triangles that are within the octagon HIJKLMNO but outside of HJLN. The area of each triangle can be found using the formula $\frac{1}{2} * a * b * \sin(135) = \frac{1}{2} * 4 * \frac{\sqrt{2}}{2} = \sqrt{2}$. The area of all four triangles is $4\sqrt{2}$. We subtract this from the entire area of the octagon to get $4\sqrt{2} + 8$
C: 2*8 =16
D: Let the side length of the square be r. By B, we know that $r^2 = 4\sqrt{2} + 8$. The perimeter is $4r$. Note that the final question includes D^2, so we can just compute 16($4\sqrt{2} + 8$) to find D^2.

$A + B + C + D^2 = 160 + 76\sqrt{2}$.

5. $\mathbf{26/3}$

A: Based on the ratio of the volume of the smaller cone to the volume of the frustum, we know that the ratio of the volume of the smaller cut cone to the original cone is $\frac{8}{27} = \left(\frac{2}{3}\right)^3$. Recall that volume ratios for similar 3D figures are cubes of side length ratios. Therefore, the similarity ratio between the smaller and original cone is $\frac{2}{3}$.

B: If we take a cross section of the sphere where the triangle lies, we have a 13-13-10 triangle circumscribed about a circle of radius r. Using the formula $r(semiperimeter) = area$, we find that $r = \frac{10}{3}$. Thus, we have the following picture:



By Pythagorean theorem, we have that $\left(\frac{26}{3}\right)^2 = \left(\frac{10}{3}\right)^2 + d^2$. Thus, $d = 8$. B=8

A + B = 2/3 + 8 = $\mathbf{\frac{26}{3}}$.

6. **-4**

Solution 1: We are told that the roots are rational. The roots can easily be found using the *p/q* factor list for the first and last exponents. $(r_1, r_2, r_3, r_4) = (-\frac{7}{2}, -3, 1, \frac{3}{2})$. A, B, C, and D can then be computed explicitly.

Solution 2: (Vieta's)

A: By the Fundamental Theorem of Algebra, $f(2) = 4(2 - r_1)(2 - r_2)(2 - r_3)(2 - r_4)$. Plugging in 2 for *x*, we find $A = \frac{55}{4}$.

B: This is just a simplified version of $\frac{sum\ of\ roots}{product\ of\ roots} = -\frac{16}{63}$.

C: Note that $r_1 + r_2 + r_3 + r_4 = -4$. We can rewrite the expression as $(-4 - r_4)(-4 - r_3)(-4 - r_2)(-4 - r_1)$. Like part A, we use the Fundamental Theorem of Algebra to finish this question. $C = (-4 - r_1)(-4 - r_2)(-4 - r_3)(-4 - r_4) = \frac{f(-4)}{4} = \frac{55}{4}$.

D: By Vieta's Formulas, the product of the roots is $\frac{63}{4}$.

$\frac{ABD}{C} = \frac{55/4*(-16)/63*63/4}{55/4}$.

7. **38**

Firstly, note that g(x) factors as follows:

$$g(x) = \frac{((x-1)^2(x-4)(x-5))}{(x-4)(x-1)(x+3)}$$

A: Using critical values, we see that *g(x)* > 0 for (-3, 1) and (5, ∞). Thus, the integers that satisfy the inequality that are less than 10 are {-2, -1, 0, 6, 7, 8, 9}. A= 27

B: 1, 4 are removable points of discontinuity (holes) in *g(x)*. B=1+4=5

C: After cancelling out holes, use polynomial division to solve for the slant asymptote of *g(x)*. The only slant asymptote is y = x – 9. Therefore, C = 9

D: The only vertical asymptote of *g(x)* is x=-3. D= -3

27+5+9-3=38

8. **1263**

A. Let $2^x = k$. We then have, $k^2 - 9k + 18 = 0$. Factor as $(k-3)(k-6) = 0$. Because $2^x = k$, $A = \lfloor \log_2 3 + \log_2 6 \rfloor = \lfloor \log_2 18 \rfloor = 4$.

B. Consider the following domains $(-\infty, -1], [-1, 3],$ and $[3, \infty)$.

In the first domain, we have the line: $y = 3 - x + 2(-x-1) + 3, slope = -3$
In the second domain, we have the line: $y = 3 - x + 2(x+1) + 3,\ slope = 1$
In the third domain, we have the line: $y = x - 3 + 2(x+1) + 3, slope = 3$
Thus, B= 9

C. McKayla will use the side of the wall as one side of the rectangular play pen such that the play pen will have dimensions of *x* and 100-2*x*. The formula for the area is *(base)(height)* or (100-2*x*)(*x*). To maximize this quadratic expression, we can complete the square:
$-2(x^2 - 50x + 625) + 1250$. Therefore, the maximum area of the play pen is 1250.

A + B + C = 4 + 9 + 1250 = 1263

9. **157**

Using minors and/or row operations, we find that the determinant of this matrix is -12.

A. Using minors and/or row operations, we find that the determinant of this matrix is -12. We will look at the third column because there are 2 zeroes in that column. The determinant is 2 * (determinant of its minor) + 1 * (determinant of its minor) = -12
B. The result of this multiplication is another 4x1 matrix. Multiply the row of M by the columns of R. The first element of M*R = 1*1 + 5*1 = 6. The second element is 3*1 +

3*1 = 6. The third element is 2*1 + 0*1 = 2. The last element is -1*1 + 1*1 =0. Therefore the sum of the elements in the resulting matrix is 6+6+2+0=14

C. The determinant of M^2 is just $\left||M|\right|^2$ = 144.

D. Multiply the first row of M by the first column of M to find the element in the first row and first column of K. D =1 * 1 + 3 * 1 + 2 * 5 +(-1)*3 = 11.

-12 + 14 + 144 + 11 = 157

10. **-14**

A. If the 7th and 15th term are equal, this means that $\binom{N}{6} = \binom{N}{14}$. Therefore, N = 6 + 14 = 20. The second term in the binomial expansion of $(x-1)^{20}$ is $\binom{20}{1}(x^1)(-1)^{19} = -20x$. Therefore, A = -20.

B. Let $x = \sqrt[3]{M} + \sqrt[3]{N}$. Then by binomial expansion ` , $x^3 = M + N + 3x\sqrt[3]{MN}$. We would like to find M and N such that the coefficients match the polynomial $x^3 - 18x - 30$. Thus, we have $M + N = 30$ and MN = 216. Solve to find that $M = 12, N = 18$. Therefore, $|\mathrm{M} - \mathrm{N}| = 6$

C. Consider the block of four consecutive terms:

$$2 + 3i - 4 - 5i = -2 - 2i$$
$$6 + 7i - 8 - 9i = -2 - 2i$$
$$10 + 11i - 12 - 13i = -2 - 2i$$

This pattern continues through 2021:

$$2018 + 2019i - 2020 - 2021i = -2 - 2i$$
$$2022 + 2023i - 2024 - 2025i = -2 - 2i$$

The number of (-2-2i) terms is 506. 506(-2-2i) = -1012 – 1012i. Thus, C = 0

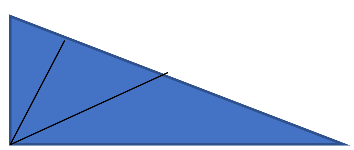
A + B + 0 = -20 + 6 + 0= -14

11. **$7\sqrt{5}/4$**

A. Because QR = 2* MR, the area of PMR is half that of PQR. Similarly, because PR = 2*PO, the area of PMO is half that of PMR. Therefore, the area of PMO is ¼ that of PQR

B. Note that the locus of points that are equidistant from BH and BS is the angle bisector of ∠HBS. By the Angle Bisector Theorem, M divides HS into segments in ratio 7:9. Therefore, HK= 7* 8/16 = 7/2

C. We have the following picture:



The altitude is of length 4 (1/2*12*h=24). The median is of length 12/2 = 6. We have a right triangle AKR where the median AK is the hypotenuse, and the altitude AR is a leg of length 4. Using the Pythagorean Theorem, we have that KR = $2\sqrt{5}$.

ABC=**$7\sqrt{5}/4$**

12. **11**

A. Recall that $\log_a b = (\log a)/(\log b)$ by difference of bases. Convert each of the logarithms in the sum into this form. The expression simplifies to 2.

B. The equation factors into $(2^x - 2)^2 = 0$. Thus, $2^x = 2$, or $x = 1$. Thus, B=1

C: Because we are adding logarithms of the same base, we can multiply *r*, *s*, and *t*. We want to find $\log_2 rst$. By Vieta's formulas, *rst* = -last/first = -(-96)/6 = 16. Thus, C=4

D: $\sqrt{7+2\sqrt{6}} = \sqrt{6}+1$, as $\left(\sqrt{6}+1\right)^2 = 6+1+2\sqrt{6}$. Apply the same idea to find $\sqrt{15-6\sqrt{6}} = 3-\sqrt{6}$. Thus, the sum is 3 + 1 = 4.

2 + 1 + 4 + 4 = 11

13. $\mathbf{12\sqrt{3}}$

A. We can rewrite the parabola into standard form as 3/2(y-14/3)=(x^2 +4x +4). This is a vertical parabola centered at (-2, 14/3) with focal radius (p) equal to 3/8. The length of the latus rectum is 4p = 3/2.
B. Eccentricity is c/a and the length of the latus rectum is $\frac{2b^2}{a}$. Also, recall that $a^2 - b^2 = c^2$. Use these three equations and the given information to find that $a = 2, b = \sqrt{3}$, c =1. Thus, the ellipse has an area of $2\pi\sqrt{3}$. Because B$\pi = the\ area$, then B = $2\sqrt{3}$
C. Complete the square

$$3(x-1)^2 + 5(y-2)^2 = 0$$

This is a point, so there is only one set of real numbers (x,y) that satisfy the equation.

D. We can use the formula for shortest distance between a point and a line. $dist = Aa + Bb - C|/(\sqrt{A^2+B^2})) = 20/5$ = 4

3/2 * $2\sqrt{3}$ * 1 * 4 = $12\sqrt{3}$

14. **196**

Circle W is a circle centered at (5,-3) with radius 5. Circle Z is centered at (10,7) with radius $5\sqrt{2}$.
A: Draw a picture! P is inside of Z. The minimum distance between P and Z is the radius minus the distance between P and the center = $5\sqrt{2} - 4\sqrt{2} = \sqrt{2}$.
B: Solve for intersection by setting equations for circle W and circle Z equal.
$x^2 - 10x + 9 + y^2 + 6y = x^2 - 20x + y^2 - 14y + 99$. Simplify to $x + 2y = 9$. Thus, B= 1+2-9 = -6.

C: Use the distance formula between (10,7) and (5,-3). C= $\underline{5\sqrt{5}.}$

D: The area of Circle W is $(5)^2\pi$. The area of Circle Z is $\left(5\sqrt{2}\right)^2\pi$. Thus, the sum of the areas is 75π. Thus, $\mathrm{D} = 75$

2 + -6 + 125 +75 = 196