2021 Theta School Bowl
Mu Alpha Theta National Convention

0: 12
1: 3800
2: 63
3: 130
4: 19
5: 20
6: 13
7: 70
8: 47040
9: 1234
10: 122
11: 2190
12: 233
13: 4
14: 2

Question 0

$A=8!,\ B=\dfrac{8!}{3!},\ C=5,\ D=1$

The five Platonic solids are tetrahedron, cube, octahedron, icosahedron, and dodecahedron.
The eccentricity of a non-degenerate parabola is 1.

$\dfrac{A}{B}+C+D=6+5+1=12$

Question 1

$A=50,\ B=3744,\ C=6$

A: The triangle is a 6-8-10 right triangle. The area of the square is $\dfrac{d^2}{2}=\dfrac{100}{2}=50.$

B: Find the two slant heights by using a right triangle with legs 10 and 24 and a right triangle with legs 32 and 24. The corresponding slant heights are 26 and 40. The total surface area is $(20)(64)+2\left[\frac{1}{2}(20)(40)\right]+2\left[\frac{1}{2}(64)(26)\right]=3744.$

C: Half a dozen is 6.

$50+3744+6=3800$

Question 2

$A = 4,\ B = 8$

$x = \sqrt[3]{7} + \sqrt[3]{49} \rightarrow x^3 = 7 + 49 + 3\sqrt[3]{7^2 \bullet 49} + 3\sqrt[3]{7 \bullet 49^2} = 56 + 3\sqrt[3]{7 \bullet 49}\left(\sqrt[3]{7} + \sqrt[3]{49}\right) = 56 + 21x \rightarrow$

$x^3 - 21x - 56 = 0$

$21 = 3^1 \bullet 7^1 \rightarrow (1+1)(1+1) = 4$ $\qquad$ $56 = 2^3 \bullet 7^1 \rightarrow (3+1)(1+1) = 8$

$4 \bullet 8 = 32 = 2^5 \rightarrow 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 63$

Question 3

$A = 3,\ B = 3,\ C = 124$

For part A:

If the base is 1: $x^2 - 5x + 5 = 1 \rightarrow (x-4)(x-1) = 0 \rightarrow x = 4, 1$

If the exponent is 0 (but not the base): $x^2 + 4x - 60 = 0 \rightarrow (x+10)(x-6) = 0 \rightarrow x = -10, 6$

If the base is –1 and the exponent is even: $x^2 - 5x + 5 = -1 \rightarrow (x-3)(x-2) = 0 \rightarrow x = 2$ only

For part B:

Let $\frac{1-x^3}{x} = k$. Then, $1 - x^3 = kx \rightarrow x^3 + kx - 1 = 0$. Since k takes on values a, b, and c, according to Vieta's theorems, $abc = 1$ and $a + b + c = 0$. $(a+b) = -c \rightarrow (a+b)^3 = (-c)^3 \rightarrow$

$a^3 + b^3 + 3ab(a+b) = -c^3 \rightarrow a^3 + b^3 + c^3 = -3ab(a+b) = -3ab(-c) = 3abc = 3(1) = 3$

For part C:

$(a+b)^2 = a^2 + b^2 + 2ab = 144 \rightarrow a^2 + b^2 = 144 - 2(10) = 124$

$3 + 3 + 124 = 130$

Question 4

$A = 41,\ B = 3300$

For part A:

$1003 - 63 = 940$ have at least one of the items. There are a total of $794 + 187 = 981$ radios and cars. That leaves 41 that have both.

For part B:

We only need values that pertain to the *View*. 40% = 4000; 5% = 500; 4% = 400; 2% = 200. $4000 - 500 - 400 + 200 = 3300$.

$3300 - 41 = 3259 \rightarrow 3 + 2 + 5 + 9 = 19$

Question 5

$A = 4,\ B = 16$

For part A:

$\dfrac{x^2+kx+1}{x^2+x+1}<2$ and $\dfrac{x^2+kx+1}{x^2+x+1}>-2$

$x^2+kx+1<2x^2+2x+2$ $\quad$ $x^2+kx+1>-2x^2-2x-2$

$x^2+(2-k)x+1>0$ $\quad$ $3x^2+(2+k)x+3>0$

Thinking of these as two parabolas, these inequalities will only be true when the discriminants are negative.

$(2-k)^2-4(1)(1)<0$ $\quad$ $(2+k)^2-4(3)(3)<0$

$(2-k+2)(2-k-2)<0$ $\quad$ $(2+k)^2-36<0$

$(4-k)(-k)<0$ $\quad$ $(2+k+6)(2+k-6)<0$

$(k-4)(k)<0$ $\quad$ $(k+8)(k-4)<0$

$(0, 4)$ $\quad$ $(-8, 4)$

The interval satisfying both is (0, 4), so $A=m+p=0+4=4$

For part B:

Since we have four distinct factors and a positive n, a number line shows us that the inequality holds for all 11 values $\left[-6, 4\right]$ and for the values $\left[5, \dfrac{n}{2}\right]$. We need four integers in the latter interval, so $\dfrac{n}{2}=8\rightarrow n=16$.

$4+16=20$

Question 6

$A=4, B=8.32$

For part A:

Simple: $1600(0.05)(2)=160$

Compound: $1600(1.05)^2-1600\rightarrow 1600\left(\dfrac{21}{20}\right)^2-1600\rightarrow 1600\left(\dfrac{441}{400}-1\right)=164$

$164-160=4$

For part B:

1600 is irrelevant. $Pe^{rt}\rightarrow P\left(1.0832\right)^1\rightarrow 8.32\%$ increase

$100\left(\dfrac{8.32}{4^3}\right)\rightarrow \dfrac{832}{64}=13$

Question 7

$A=1, B=9, C=58$

For part A:

$28=1+(n-1)(3)\rightarrow n=10$

$\dfrac{10}{2}\left[(x+1)+(x+28)\right]=155\rightarrow 2x+29=31\rightarrow x=1$

For part B:

$$S=(n-2)(180)=\frac{n}{2}\left[2(120)+(n-1)(5)\right]$$

$$180n-360=\frac{n}{2}(235+5n)\to 360n-720=5n^2+235n\to n^2-25n+144=0$$

$$\to(n-16)(n-9)=0$$

$n=9$, as $n=16$ produces an angle of $195°$.

For part C:

$$\frac{2xy}{x+y}=4.2=\frac{21}{5}\to\frac{xy}{\left(\frac{x+y}{2}\right)}=\frac{21}{5}$$

$$a=5,\ g=\sqrt{21}\to 3(5)+21=36$$

$$(x+y)^2=x^2+y^2+2xy\to 100=x^2+y^2+2(21)\to x^2+y^2=58$$

$$3(1)+9+58=70$$

Question 8

$A=10, B=60, C=10\sqrt{78.4}$

For part A:

We must use three consecutive vertices in order to get a triangle that shares two sides of the decagon. This will be a total of 10 triangles.

For part B:

We must use two consecutive vertices for the side of the triangle. The third vertex can't be adjacent to either of the vertices that "create" the side. This leaves 6 vertices as options. There are 10 sets of two consecutive vertices, giving us 60 triangles.

For part C:

Using the law of cosines, the length x of a side of the decagon is found by

$x^2=14^2+14^2-2(14)(14)(0.8)=78.4\to P=10\sqrt{78.4}$.

$$\frac{60[100(78.4)]}{10}=47040$$

Question 9

$A=2, B=1936, C=-704$

For part A:

$$C1+C2+C3\to C1\to \text{ gives } f(x)=\begin{vmatrix} 1+2x+\left(a^2+b^2+c^2\right)x & \left(1+b^2\right)x & \left(1+c^2\right)x \\ 1+2x+\left(a^2+b^2+c^2\right)x & 1+b^2x & \left(1+c^2\right)x \\ 1+2x+\left(a^2+b^2+c^2\right)x & \left(1+b^2\right)x & 1+c^2x \end{vmatrix}.$$

Substituting $a^2+b^2+c^2=-2$ gives $f(x)=\begin{vmatrix}1 & (1+b^2)x & (1+c^2)x \\ 1 & 1+b^2x & (1+c^2)x \\ 1 & (1+b^2)x & 1+c^2x\end{vmatrix}$.

$R2-R1\to R2$ and $R3-R1\to R3$ gives $f(x)=\begin{vmatrix}1 & (1+b^2)x & (1+c^2)x \\ 0 & 1-x & 0 \\ 0 & 0 & 1-x\end{vmatrix}$.

This is now a 3×3 upper triangular matrix, so the determinant is $(1)(1-x)(1-x)$, which is degree 2.

For part B:

$$B=\left|\begin{bmatrix}7 & 2 \\ 8 & -4\end{bmatrix}\begin{bmatrix}7 & 2 \\ 8 & -4\end{bmatrix}\right|=\begin{vmatrix}65 & 6 \\ 24 & 32\end{vmatrix}=1936.$$

For part C:

$$C=4^2\begin{vmatrix}7 & 2 \\ 8 & -4\end{vmatrix}=-704.$$

$$2+1936-704=1234$$

Question 10

$$A=7,\ B=2,\ C=-11,\ D=\frac{\sqrt{7}}{4}$$

For part A:

$$h=-\frac{(-72)}{2(9)}=4,\ k=-\frac{96}{2(-16)}=3\to 4+3=7$$

For part B:

$$9\left(x^2-8x+16\right)-16\left(y^2-6y+9\right)=144+144-144\to\frac{(x-4)^2}{16}-\frac{(y-3)^2}{9}=1$$

The asymptotes will pass through (4, 3) and have slopes $\pm\frac{3}{4}$. $y+3=\pm\frac{3}{4}(x-4)$ gives x-intercepts 8 and 0 and y-intercepts –6 and 0. The sum of these is 2.

For part C:

For the x-intercepts: $9x^2+72x-144=0\to x^2+8x-16=0\to(x+4)^2=16\to x=-4\pm 4\sqrt{2}$

For the y-intercepts: $-16y^2-96y-144=0\to y^2+6y+9=0\to(y+3)^2=0\to y=-3$

The sum of these is –11.

For part D:

$$9\left(x^2+8x+16\right)+16\left(y^2-6y+9\right)=144+144+144\to\frac{(x+4)^2}{48}+\frac{(y-3)^2}{27}=1$$

$c^2 = a^2 - b^2 = 48 - 27 = 21 \rightarrow e = \frac{c}{a} = \frac{\sqrt{21}}{\sqrt{48}} = \frac{\sqrt{7}}{4}$

$\frac{2^{2(2)}}{7}\left(\frac{7}{16}\right) + (-11)^2 = 1 + 121 = 122$

Question 11

$A = \frac{10}{3}, B = 117, C = 360, D = 5$

For part A:

$\frac{x}{10} + \frac{x}{5} = 1 \rightarrow \frac{x + 2x}{10} = 1 \rightarrow x = \frac{10}{3}$

For part B:

$m\angle E + m\angle H = 180 - 54 = 126°$. $m\angle IEH + m\angle IHE + m\angle EIH = 180°$.

$m\angle IEH + m\angle IHE = \frac{1}{2}(m\angle E + m\angle H) = 63°$. $m\angle EIH = 180 - 63 = 117°$.

For part C:

Draw $\overline{ER}$. This creates two cyclic quadrilaterals, where opposite angles are supplementary. $m\angle EBI + m\angle ERI = 180°; m\angle CSE + m\angle ERC = 180°$.

$m\angle IRC = m\angle ERI + m\angle ERC$, so the sum of the three angles is $180 + 180 = 360°$.

For part D:

$T = \frac{D}{R} \rightarrow \frac{120}{25 - x} = \frac{120}{25 + x}\left(\frac{3}{2}\right) \rightarrow 2(25 + x) = 3(25 - x) \rightarrow x = 5$.

$\left(\frac{10}{3}\right)(117) + (360)(5) = 390 + 1800 = 2190$

Question 12

$A = 1, B = 7, C = 196, D = 252$

For part A:

$8^8 \bmod 15 = 2^{24} \bmod 15 \rightarrow 2^4 \equiv 1 \bmod 15$, so $\left(2^4\right)^6 \equiv (1)^6 \bmod 15 = 1 \bmod 15$.

For parts B and C:

$20x + 16x = 500 \rightarrow 5x + 4y = 125$. We can see by inspection that (25, 0) is a solution. We can now decrease the x-values by 4 and increase the y-values by 5 to find all the solutions. We will get 7 solutions: (25, 0), (21, 5), (17, 10), (13, 15), (9, 20), (5, 25), and (1, 30). The sum of the x- and y-values is 196.

For part D:

There are $999 - 99 = 900$ three-digit integers. Let's find the number of three-digit integers that contain NO 5s. There are 8 possibilities for the first digit, and 9 possibilities for the second and third digits. $(8)(9)(9) = 648$. $900 - 648 = 252$ three-digit integers with no 5s.

$\frac{252}{7}+1+196=233$

Question 13

$A=4, B=12, C=9, D=3$

For part A:

$0.009423=9.423\times10^{-3}$, so the characteristic is -3. $3^7<2499<3^8$, so the characteristic is 7. The sum is 4.

For part B:

$(b^2+6b+6)(5b+6)=8b^3+5b^2+9b \to 5b^3+30b^2+30b+6b^2+36b+36=8b^3+5b^2+9b \to$

$3b^3-31b^2-57b-36=0 \to (b-12)(3b^2+5b+3)=0 \to b=12.$ The other factor has non-real roots.

For part C:

$6x=x^2+a \to x^2-6x+a=0.$ Discriminant must be 0: $36-4a=0 \to a=9.$

For part D:

$$\frac{1}{\log_8 24}+\frac{1}{3\log_{64}2+\log_{64}3}+\frac{3}{\log_3 24} \to \frac{1}{\log_8 24}+\frac{1}{\log_{64}24}+\frac{3}{\log_3 24} \to$$

$$\log_{24}8+\log_{24}64+\log_{24}27 \to \log_{24}13824=3.$$

$(4)(12)(9)(3)\bmod 17=432\bmod 17=4$

Question 14

$A=3, B=-1, C=6, D=3, E=5, F=3$

$A=(g^{-1}\circ f^{-1})(2)=g^{-1}(5)=3$

$B=(g^{-1}\circ h)(4)=g^{-1}(1)=-1$

$C=(h^{-1}\circ f\circ g^{-1})(3)=(h^{-1}\circ f)(2)=h^{-1}(-1)=6$

$D=(g\circ f^{-1})(-1)=g(2)=3$

$E=(f^{-1}\circ g^{-1})(3)=f^{-1}(2)=5$

$F=(h^{-1}\circ g^{-1}\circ f)(6)=(h^{-1}\circ g^{-1})(3)=h^{-1}(2)=3$

Vertices: $3-i, 6+3i, 5+3i$

Area: $\pm\frac{1}{2}\begin{vmatrix}3 & -1 & 1\\ 6 & 3 & 1\\ 5 & 3 & 1\end{vmatrix}=\frac{1}{2}(4)=2$