Answers: ADBCC CCAED ADAAB ADBBD CBCAE BCDAC

01) The desired area is that of an equilateral triangle with side length 6 being removed from a 60-degree wedge of a circle with the same radius. This area is $\frac{1}{6} \cdot 36\pi - \frac{6^2\sqrt{3}}{4} = 6\pi - 9\sqrt{3}$. $A + B + C = 18$.

02) By the Shoelace Theorem as the problem hints, the area of the triangle is $\frac{1}{2}\left(\begin{vmatrix} -3 & 2 \\ 8 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ -1 & 5 \end{vmatrix} + \begin{vmatrix} 4 & 0 \\ 5 & 9 \end{vmatrix} + \begin{vmatrix} 0 & -3 \\ 9 & 8 \end{vmatrix}\right) = \frac{-13+14+36+27}{2} = 32$. Subtraction of areas of triangles from an inscribing rectangle can also be used to obtain the same answer.

03) The initial surface area of the cube is $6 \cdot 3^2 = 54$. Each added unit cube covers one of its own faces and a region on the original cube of equal size but adds 5 other 1-by-1 faces for a net added area of 4. For six added unit cubes, the proportional increase in surface area is $\frac{24}{54} = \frac{4}{9} \approx 44\%$.

04) The two curves intersect at $x = \pm\sqrt{\frac{2}{3a}} = \pm k$. The distance between the curves is $2 - 3ax^2$. Integrating, $\int_{-k}^{k}(2 - 3ax^2)\,dx = \int_0^k (4 - 6ax^2)\,dx = 4x - 2ax^3\big]_0^k = 2k(2 - ak^2) = \frac{8}{3}\sqrt{\frac{2}{3a}}$. Let $c = \frac{8}{3}\sqrt{\frac{2}{3}}$. The derivative of $\frac{c}{\sqrt{a}}$ with respect to t is $-\frac{c}{2a\sqrt{a}}\frac{da}{dt}$. Substituting $a = 6$ and $\frac{da}{dt} = 2$ yields an instantaneous rate of change of $-\frac{4}{27}$.

05) The volume of the cylindrical tank is $V = \pi r^2 h = 9\pi h$. Deriving, $\frac{dV}{dt} = 9\pi\frac{dh}{dt} = -9\pi$, so water is entering the cone at a constant rate of 9π. For the cone, $r = \frac{2h}{3}$, so $V = \frac{\pi r^2 h}{3} = \frac{4\pi h^3}{27}$, so $\frac{dV}{dt} = \frac{4\pi h^2}{9}\frac{dh}{dt}$. The cylindrical tank has lost half of its volume, which is $\frac{9\pi \cdot 8}{2} = 36\pi$. Solving $\frac{4\pi h^3}{27} = 36\pi$ yields $h = 3\sqrt[3]{9}$. Substituting this and $\frac{dV}{dt} = 9\pi$ into our equation gives $\frac{dh}{dt} = \frac{\sqrt[3]{9}}{4}$.

06) For all $n \geq 0$, Corey has 2^n dice, each with volume $\left(\frac{2}{5}\right)^{3n}$, making for a total volume of $\left(\frac{16}{125}\right)^n$. The sum of the infinite geometric series with this ratio and initial term 1 is $\frac{125}{109}$. $125 + 109 = 234$.

07) Polynomial long division yields that the function is equal to $x^6 - x^4 + x^2 - 1 + \frac{2}{x^2+1}$. Integrating this from 0 to 1 yields a value of $\frac{1}{7} - \frac{1}{5} + \frac{1}{3} - 1 + 2 \cdot \frac{\pi}{4} = \frac{\pi}{2} - \frac{76}{105}$. $A + B + C = 183$, which leaves a remainder of 3 when divided by 9.

08) The radii of two tangent circles would combine to form each side of the triangle. Solving the system $a + b = 20$, $a + c = 22$, and $b + c = 24$ yields $a = 9$, $b = 11$, and $c = 13$. The sum of the areas of the circles with these radii is $81\pi + 121\pi + 169\pi = 371\pi$.

09) $f'(x) = 3x^2 - 6x + 3$ and $f''(x) = 6x - 6$. $f'(x) = f''(x)$ at $x = 1$ and $x = 3$. Integrating, $\int_1^3(-3x^2 + 12x - 9)\,dx = -x^3 + 6x^2 - 9x\big]_1^3 = 0 + 4 = 4$.

10) In a triangle, $1 + \frac{r}{R} = \cos A + \cos B + \cos C$, so $\frac{15}{13} = 3\cos B$ and $\cos B = \frac{5}{13}$. By the Projection Law, $a = b\cos C + c\cos B$ and $c = a\cos B + b\cos A$, so adding, $a + c = b(\cos C + \cos A) + (a + c)\cos B$. Since $\cos C + \cos A = 2\cos B$, $a + c = (a + 2b + c)\cos B = \frac{5(a+2b+c)}{13}$,

Simplifying, $a + c = \frac{5b}{4}$ and $s = \frac{9b}{8}$. $rR = \frac{abc}{4s}$, so $26 = \frac{2ac}{9}$ and $ac = 117$. $\sin B = \frac{12}{13}$, so the area of the triangle is $\frac{ac}{2}\sin B = 54$.

11) The area of a triangle is $\frac{ab\sin C}{2}$, so we know $\frac{a'b\sin C + ab'\sin C + abC'\cos C}{2}$ is constant. Plugging in values, this yields $-9 - 9 + 27\sqrt{3}C' = 0$, so $C' = \frac{2\sqrt{3}}{9}$.

12) $\int_{-\sqrt{15}}^{\sqrt{15}}((16 - x^2)^2 - 1^2)\,dx = 2\int_0^{\sqrt{15}}(x^4 - 32x^2 + 255)\,dx = \frac{x^5}{5} - \frac{32x^3}{3} + 255x\Big]_0^{\sqrt{15}} = 280\sqrt{15}$.

13) The graphs of these polar curves intersect when $\theta = \frac{\pi}{3}$. $\frac{1}{2}\int_0^{\pi/3}\sin^2\theta\;d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2}\sin^2 2\theta\;d\theta = \frac{1}{2}\int_0^{\pi/3}\sin^2\theta\;d\theta + \frac{1}{4}\int_{2\pi/3}^{\pi}\sin^2\theta\;d\theta$. Noting that the integral of $\sin^2\theta$ is $\frac{2\theta - \sin 2\theta}{4}$, these integrals are equal to $\frac{2\theta - \sin\theta}{8}\Big]_0^{\pi/3} + \frac{2\theta - \sin 2\theta}{16}\Big]_{2\pi/3}^{\pi} = \left(\frac{\pi}{12} - \frac{\sqrt{3}}{16}\right) + \left(\frac{\pi}{8} - \left(\frac{\pi}{12} + \frac{\sqrt{3}}{32}\right)\right) = \frac{\pi}{8} - \frac{3\sqrt{3}}{32}$.

14) The base of a regular tetrahedron is an equilateral triangle with area $\frac{s^2\sqrt{3}}{4}$. The altitude has length $\frac{s\sqrt{3}}{2}$. The fourth vertex is above the center of the base, which is $\frac{1}{3}$ of the way up an altitude of the triangle. The height can be found by solving $\left(\frac{s}{\sqrt{3}}\right)^2 + h^2 = s^2$, so $h^2 = \frac{2s^2}{3}$ and $h = \frac{s\sqrt{6}}{3}$. Plugging into $V = \frac{Bh}{3}$ yields $V = \frac{s^3\sqrt{2}}{12}$.

15) $h = \frac{3r}{2}$, so $V = \frac{\pi r^2 h}{3} = \frac{\pi r^3}{2}$. Deriving, $\frac{dV}{dt} = \frac{3\pi r^2}{2}\frac{dr}{dt}$. Solving $\frac{\pi r^3}{2} = 4\pi$ gives $r = 2$. Substituting in values, $12\pi = 6\pi\frac{dr}{dt}$ and $\frac{dr}{dt} = 2$. Diameter is twice radius, so $\frac{dd}{dt} = 4$.

16) This shape is the rotation of the region bounded by $y = \sqrt{16 - x^2}$ and the line $y = 2$ over the x-axis. $\pi\int_{-2\sqrt{3}}^{2\sqrt{3}}(12 - x^2)\,dx = 2\pi\int_0^{2\sqrt{3}}(12 - x^2)\,dx = 24\pi x - \frac{2\pi x^3}{3}\Big]_0^{2\sqrt{3}} = 32\pi\sqrt{3}$.

17) Multiply by 2 to get a Riemann sum. $\frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2}{n}\cdot\left(3 - \left(1 + \frac{2k}{n}\right)^2\right) = \frac{1}{2}\int_1^3(3 - x^2)\,dx = \frac{3x}{2} - \frac{x^3}{6}\Big]_1^3 = -\frac{4}{3}$.

18) The slope of the tangent line to the unit circle for a given angle θ is equal to $-\cot\theta$. The area of the triangle is $\frac{\tan\theta}{2}$, and the area of the sector is $\frac{\theta}{2}$, so the area of the desired region is $\frac{\tan\theta - \theta}{2}$. The rate at which this changes with respect to θ is $\frac{\sec^2\theta - 1}{2}\frac{d\theta}{dt}$. The particle moves $\frac{\pi}{2}$ radians in 5 seconds for $\frac{d\theta}{dt} = \frac{\pi}{10}$. Since $\sec\theta = \frac{5}{4}$, the change of area is $\frac{9\pi}{320}$.

19) $\frac{P+4}{P-4} = A - 1$ can be rearranged to $AP - 2P - 4A = 0$. Simon's Favorite Factoring Trick can be used to create the equation $(P - 4)(A - 2) = 8$. $P - 4$ and $A - 2$ are integers, and P and A must both be positive. The possible solutions (P, A) are (5,10), (6,6), (8,4), and (12,3). A rectangle of perimeter P can have a maximum area of $\left(\frac{P}{4}\right)^2$. Only the last 2 solutions are valid rectangles, so the sum of the possible values of P is $8 + 12 = 20$.

20) Fully calculating each approximation is not necessary; Rohan and Christina both use the same five regions with the middle points and only differ in which region they add to the end. Rohan uses the rectangle with height 6 while Christina uses the rectangle with height -4. Since the widths of the rectangles are both 2, the difference in area is 20.

21) Solving for y yields $y = \sqrt[4]{1 - \frac{x^2}{16}}$, the half-base of the solid. The volume of the solid is $\frac{\sqrt{3}}{4}\int_{-4}^{4} 4\sqrt{1-\frac{x^2}{16}}\ dx = \frac{\sqrt{3}}{2}\int_0^4 \sqrt{16-x^2}\ dx = \frac{\sqrt{3}}{2}\cdot 4\pi = 2\pi\sqrt{3}$.

22) $\frac{dx}{d\theta} = -3\cos^2\theta\sin\theta$ and $\frac{dy}{d\theta} = 3\sin^2\theta\cos\theta$. $ds = \sqrt{9\cos^4\theta\sin^2\theta + 9\sin^4\theta\cos^2\theta} = 3\sin\theta\cos\theta\sqrt{\cos^2\theta+\sin^2\theta} = 3\sin\theta\cos\theta$. $\int_0^{\pi/2} 2\pi y\ ds = 6\pi\int_0^{\pi/2}\sin^4\theta\cos\theta\ d\theta = 6\pi\int_0^1 u^4\ du = \frac{6\pi}{5}$.

23) Set $x' = x + 2y$ and $y' = 2x - y$. Recognizing this as looking similar to a rotation matrix, rewrite these equations as $x' = \sqrt{5}\left(\frac{x}{\sqrt{5}} + \frac{2y}{\sqrt{5}}\right)$ and $y' = \sqrt{5}\left(\frac{2x}{\sqrt{5}} - \frac{y}{\sqrt{5}}\right)$. Setting $\sin\theta = \frac{2}{\sqrt{5}}$ and $\cos\theta = \frac{1}{\sqrt{5}}$ so that $x' = \sqrt{5}(x\cos\theta + y\sin\theta)$ and $y' = \sqrt{5}(x\sin\theta - y\cos\theta)$ yields that a rotation of θ and scaling the dimensions up by a factor of $\sqrt{5}$ results in $|x'| + |y'| = 6$, a square with diagonal length 12 and whose area is therefore 72. Scaling the area down by $\sqrt{5}^2$ to obtain the original graph yields an area of $\frac{72}{5}$.

24) The graph of $r = 2\sin 3\theta$ has 3 petals, each with volume $\frac{1}{2}\int_0^{\pi/3} 4\sin^2 3\theta\ d\theta = \frac{1}{3}\int_0^{\pi} 2\sin^2 u\ du$. Using the double angle formula, this is $\frac{1}{3}\int_0^{\pi}(1-\cos 2u)\ du = \frac{u}{3} - \frac{\sin 2u}{6}\Big]_0^{\pi} = \frac{\pi}{3}$. The graph of $r = 3\sin 4\theta$ has 8 petals, each with volume $\frac{1}{2}\int_0^{\pi/4} 9\sin^2 4\theta\ d\theta = \frac{9}{16}\int_0^{\pi} 2\sin^2 u\ du$. Using the double angle formula, this is $\frac{9}{16}\int_0^{\pi}(1-\cos 2u)\ du = \frac{9u}{16} - \frac{9\sin 2u}{32}\Big]_0^{\pi} = \frac{9\pi}{16}$. The total area of all petals is $\frac{11\pi}{2}$ and there are 11 total petals, so the expected area of a single petal is $\frac{\pi}{2}$. $1 + 2 = 3$.

25) $y = x + \sin x$ and its inverse intersect at $x = 0$ and $x = \pi$. The functions are symmetric about the point (π, π). The area of the whole region is twice the area of the region bounded by $y = x + \sin x$ and its inverse between $x = 0$ and $x = \pi$. Consider the square with opposite corners at the origin and (π, π). Twice the area under the graph of $y = x + \sin x$ subtracted from the area of the square gives the negative of the area between $y = x + \sin x$ and its inverse by the Principle of Inclusion and Exclusion. We have $\int_0^{\pi}(x + \sin x)\ dx = \frac{x^2}{2} - \cos x\Big]_0^{\pi} = \frac{\pi^2}{2} - 2$. The area bounded by $y = x + \sin x$ and its inverse in the square is $2\left(\frac{\pi^2}{2} - 2\right) - \pi^2 = 4$, so the total area of the region is 8.

26) The line tangent to the graph of $y = x^2$ at (a, a^2) has slope $2a$ and thus equation $y = 2ax - a^2$, or $-2ax + y + a^2 = 0$. The distance from $(0,4)$ to this line is $\frac{4+a^2}{\sqrt{4a^2+1}}$. The derivative of this is $\frac{2a(2a^2-7)}{(4a^2+1)^{3/2}}$, which equals 0 at $a = \sqrt{\frac{7}{2}}$. Evaluated, this corresponds to a minimum distance of $\frac{\sqrt{15}}{2}$. By the Theorem of Pappus, the minimum volume is $\pi\cdot 2\pi\cdot\frac{\sqrt{15}}{2} = \pi^2\sqrt{15}$.

27) The lines intersect at the points $(3,5)$, $(1,4)$, and $(6,1)$. The area of the triangle whose vertices are these points is $\frac{1}{2}\begin{vmatrix} 3 & 5 & 1 \\ 1 & 4 & 1 \\ 6 & 1 & 1 \end{vmatrix} = \frac{11}{2}$.

28) The volume rotating about the x-axis is $\pi\int_0^{\pi/2} x^2\cos^2 x\ dx = \frac{\pi}{2}\int_0^{\pi/2} x^2\ dx + \frac{\pi}{2}\int_0^{\pi/2} x^2 \cos 2x\ dx$ The first of these integrals is equal to $\frac{\pi^4}{48}$. The second integral can be solved by parts, equaling $\left.\frac{\pi x^2 \sin 2x}{4}\right]_0^{\pi/2} - \frac{\pi}{2}\int_0^{\pi/2} x \sin 2x\ dx = \left.\frac{\pi x \cos 2x}{4}\right]_0^{\pi/2} - \frac{\pi}{4}\int_0^{\pi/2} \cos 2x\ dx = -\frac{\pi^2}{8}$. Thus, the volume rotating about the x-axis is $\frac{\pi^4}{48} - \frac{\pi^2}{8}$. The volume rotating about the y-axis is $2\pi\int_0^{\pi/2} x^2 \cos x\ dx$. This can be solved by parts, equaling $2\pi x^2 \sin x]_0^{\pi/2} - 4\pi\int_0^{\pi/2} x \sin x\ dx = \frac{\pi^3}{2} - [4\pi x \cos x]_0^{\pi/2} - 4\pi\int_0^{\pi/2} \cos x\ dx = \frac{\pi^3}{2} - 4\pi$. The sum of the areas is $\frac{\pi^4}{48} + \frac{\pi^3}{2} - \frac{\pi^2}{8} - 4x$, so $f(x) = \frac{x^4}{48} + \frac{x^3}{2} - \frac{x^2}{8} - 4x$. $f(12)$ can be calculated with the synthetic division.

1/48	1/2	−1/8	−4	0
	1/4	9	213/2	1230
1/48	3/4	71/8	205/2	1230

The remainder when 1230 is divided by 19 is 14.

29) The limaçon's inner loop is where $2 + 4\cos\theta = 0$, between $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$. Integrating, $\frac{1}{2}\int_{2\pi/3}^{4\pi/3}(2 + 4\cos\theta)^2\ d\theta = \int_{2\pi/3}^{4\pi/3}(8\cos^2\theta + 8\cos\theta + 2)\ d\theta = \int_{2\pi/3}^{4\pi/3}(4\cos 2\theta + 8\cos\theta + 6)\ d\theta = 2\sin 2\theta + 8\sin\theta + 6\theta]_{2\pi/3}^{4\pi/3} = \left(8\pi - 3\sqrt{3}\right) - \left(4\pi + 3\sqrt{3}\right) = 4\pi - 6\sqrt{3}$.

30) The foci of the ellipse are at $(1, -2i)$ and $(-3, i)$, which are a distance of 5 apart. The focal radius is $\frac{5}{2}$. The major axis is 6, so the semimajor axis has length 3. Solving $\left(\frac{5}{2}\right)^2 = 3^2 - r_2^2$ gives the length of the semiminor axis, $r_2 = \frac{\sqrt{11}}{2}$. The area of the ellipse is $\frac{3\pi\sqrt{11}}{2}$, so $A + B + C = 16$.