1. C
2. D
3. E
4. C
5. C
6. D
7. D
8. B
9. A
10. E
11. A
12. E
13. C
14. A
15. A
16. E
17. D
18. B
19. A
20. D
21. C
22. D
23. C
24. B
25. C
26. E
27. D
28. A
29. B
30. D

1. C By Principle of Inclusion-Exclusion, $210 + 140 - 70 = \boxed{280}$
2. D Taking the 404^{th} root of each number we find that $4^{1616} > 3^{1212} > 2^{2020} > \boxed{5^{808}}$
3. E Since $2020 = 2^2 \cdot 5 \cdot 101$, we have $3 \cdot 2 \cdot 2 = \boxed{12}$ factors
4. C Since $21600 = 2^5 \cdot 3^3 \cdot 5^2$, we have $(1+4+16)(1+9)(1+26) = \boxed{5460}$
5. C n must be in the form p^3 or $p \cdot q$. Calculating, we find $p = 2{,}3$ in the first case and $(p,q) = (2,3), (2,5), (2,7), (2,11), (2,13), (3,5), (3,7)$ in the second case for a total of $\boxed{9}$ possibilities.
6. D We have $225 = 3^2 \cdot 5^2$. For each factor a of 225 there exists a unique factor b of 225 such that $a \cdot b = 225$, except when $a = 15$. Since there are 8 such values of a, the product of all factors is $225^{\frac{8}{2}} \cdot 15 = \boxed{15^9}$
7. D $$n^2 + 3n + 2 = 49 + 7 \Rightarrow n = 6$$
8. B $$-11 = (-2)^5 + (-2)^4 + (-2)^2 + (-2)^0 \Rightarrow \boxed{4}$$
9. A Since 7 leaves a remainder of -1 when divided by 8, we may take the alternating sum of the digits to determine whether it is divisible by 7 (similar to determining divisibility for 11 in base 10).
10. E Each zero in base 12 appears with each factor of $12 = 2^2 \cdot 3$. We find that there are $\left\lfloor\frac{100}{2}\right\rfloor + \left\lfloor\frac{100}{4}\right\rfloor + \left\lfloor\frac{100}{8}\right\rfloor + \left\lfloor\frac{100}{16}\right\rfloor + \left\lfloor\frac{100}{32}\right\rfloor + \left\lfloor\frac{100}{64}\right\rfloor = 97$ powers of 2 and $\left\lfloor\frac{100}{3}\right\rfloor + \left\lfloor\frac{100}{9}\right\rfloor + \left\lfloor\frac{100}{27}\right\rfloor + \left\lfloor\frac{100}{81}\right\rfloor = 48$ powers of 3 in 100! Since $\frac{97}{2} > 48$, there are $\boxed{48}$ zeros.
11. A Let $f(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \cdots + a_1 \cdot x + a_0$. The first condition tells us that the sum of coefficients of f is less than 6. We also know that, $$f(6) = a_n \cdot 6^n + a_{n-1} \cdot 6^{n-1} + \cdots + a_1 \cdot 6 + a_0 = 1766$$ Since $a_n, a_{n-1}, \dots, a_0 < 6$, a_i are precisely the digits in the base 6 expansion of 1766. Calculating we find $1766_{10} = 12102_6$. Then $f(10) = \boxed{12102}$
12. E Starting from the solution $(0,40)$, we see that all solutions are in the form $(5x, 40 - 7x)$. In order to minimize $m + n$ we need to minimize $5x + 40 - 7x = 40 - 2x$, with the constraint $40 - 7x > 0$. We take $x = 5$ which gives the solution $(25,5)$. So, the minimum possible value is $25 + 5 = \boxed{30}$
13. C Since $12 = 2^2 \cdot 3$, if 2^a and 3^b divide x, where a, b are maximized, then $5 | 3a + 2$ and $5 | 3b + 1$. We find that the smallest possible values for a, b are $a = 1, b = 3$. This gives $x = 54, y = 18$, so $54 + 18 = \boxed{72}$
14. A Calculating, we find that the units digit cycles every 4 by the exponent. $7^0 = \boxed{1}$
15. A Calculating, we find that $4^4 = 256$ leaves a remainder of 1 when divided by 17. Thus, $4^{2020} = 4^{4 \cdot 505}$ leaves a remainder of 1 as well.
16. E Multiplying through by ab and factoring we get $$(a-7)(b-3) = 21$$ There are 4 positive factors of 21, so we set $a - 7$ equal to the 8 positive and negative factors. However, there is 1 extraneous solution given, $a = 0, b = 0$. In total, there are $\boxed{7}$ solutions.
17. D Note the smallest value of xyz possible is achieved by $x = 3, y = 4, z = 5$, with $xyz = 60$. Examining the equation mod 3, mod 4, mod 5, we see that each product xyz is necessarily divisible by 60.
18. B Let $0 \le r < 16$ be the remainder of x when divided by 16. Then we know that $16 | 3r - 7$. By examination we see $r = \boxed{13}$

19. A We observe that y, z must have the same parity. Let $y = 2y_1$ and $z = 2z_1$. Then the equation $x + y_1 + z_1 = 4$ has $\binom{6}{2} = 15$ solutions. Let $y = 2y_1 + 1$ and $z = 2z_1 + 1$. Then the equation $x + y_1 + z_1 = 3$ has $\binom{5}{2} = 10$ solutions. In total, there are $15 + 10 = \boxed{25}$ solutions.

20. D Factoring by cubes,
$$2^{18} - 1 = (2^6 - 1)(2^{12} + 2^6 + 1) = 63 \cdot 4161 = 3^3 \cdot 7 \cdot 19 \cdot \boxed{73}$$

21. C
$$\phi(84) = 84\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{7}\right) = \boxed{24}$$

22. D First, observe that if n has 3 or more prime factors then $\phi(n) \geq 8$. For any prime $p|n$ we must have $p - 1|6$. This implies that the only possible factors of n are 2,3,7. If $n = p_1^{e_1}$, then we have $\phi(n) = p_1^{e_1-1}(p_1 - 1)$, which gives $n = 7{,}9$. If $n = p_1^{e_1}p_2^{e_2}$, then $\phi(n) = p_1^{e_1-1}(p_1 - 1)p_1^{e_2-1}(p_2 - 1)$, which gives $n = 14{,}18$. The sum of all such n is $7 + 9 + 14 + 18 = 48$

23. C Observe that $\gcd(m, 500 - m) = \gcd(m, 500)$. So the number of solutions is $\phi(500) = \boxed{200}$

24. B Let $n = p_1^{e_1}p_2^{e_2} \dots p_n^{e_n}$. Since
$$\frac{\phi(n)}{n} = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\dots\left(1 - \frac{1}{p_n}\right)$$
$\frac{\phi(n)}{n}$ is minimized by selecting as many prime factors of n as possible. Then, $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$ minimizes our value. The sum of digits is $2 + 3 + 1 + 0 = \boxed{6}$

25. C Using the Euclidean Algorithm,
$$\gcd(73824{,}6432) = \gcd(3072{,}6432) = \gcd(3072{,}288) = \gcd(192{,}288) = 96$$

26. E Let $d|6n + 5, 9n + 8$. Then, $d|2(9n + 8) - 3(6n + 5) \Rightarrow d|1$. So, there are no such values of n.

27. D
$$d|3n^2 + 1 - 3(n^2 + 2n + 4) \Rightarrow d|6n + 11$$
$$d|(6n + 11)^2 - 12(3n^2 + 1) \Rightarrow d|132n + 109$$
$$d|22(6n + 11) - (132n + 109) \Rightarrow d|133$$
$$133 = 7 \cdot 19 \Rightarrow (1 + 7)(1 + 19) = \boxed{160}$$

28. A We may approximate $x(x + 2)(x + 4)$ by x^3. Then, notice that $70^3 < 438672 < 80^3$. Examining the units digit of x, we conclude that $x = \boxed{74}$

29. B Looking at the parity of the equation, we conclude that $q = 2$. Then,
$$2pr = (p + 7)(r + 6) \Rightarrow (p - 7)(r - 6) = 84$$
Observe that $p - 7$ must contain all powers of 2 of 84. Trying cases, we find $p = 19, r = 13$. Then, $pq + qr + pr = 19 \cdot 2 + 2 \cdot 13 + 13 \cdot 19 = \boxed{311}$

30. D Notice that $8^n + n = (2^n + n)4^n - (2^n + n)2^n \cdot n + (2^n + n)n^2 - n^3 + n$. Then, $2^n + n|8^n + n \to 2^n + n| - n^3 + n$. This implies that $2^n + n \leq |-n^3 + n|$. This inequality only holds for $n \leq 9$. Inspecting each option (using divisibility rules for larger values of n), we see that only $n = 1{,}2{,}4{,}6$ satisfy our condition, so our sum is $1 + 2 + 4 + 6 = \boxed{13}$