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|---|---|---|---|---|---|
| 1 | C | 11 | D | 21 | B |
| 2 | D | 12 | B | 22 | D |
| 3 | A | 13 | C | 23 | C |
| 4 | A | 14 | B | 24 | A |
| 5 | A | 15 | B | 25 | C |
| 6 | E | 16 | D | 26 | D |
| 7 | A | 17 | C | 27 | C |
| 8 | D | 18 | A | 28 | B |
| 9 | B | 19 | C | 29 | C |
| 10 | B | 20 | D | 30 | A |

1) $2 \cdot 2 - 2 \cdot 0 = 4 = 0 = 4.$ $\boxed{\text{C}}$

2) The solution to $3x + 5y = 4$, $7x - y = 3$ is $\left(\frac{1}{2}, \frac{1}{2}\right)$. $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. $\boxed{\text{D}}$

3) $M$ is a 2-by-4 matrix, and $N$ is a 3-by-2 matrix. For A), $M^T$ is a 4-by-2 matrix, and $N^T$ is a 2-by-3 matrix, so $MM^T$ and $N^TN$ are 2-by-2 and 2-by-2 matrices respectively. Since they have the same dimension, they can be added. We will now confirm that the other choices are false. For B), while $NM$ exists, it is a 3-by-4 matrix and thus cannot be multiplied by itself. For C), a 2-by-4 matrix cannot be added to a 2-by-3 matrix. For D), a matrix can only be an exponent to a constant if it is square. $\boxed{\text{A}}$

4) The matrix is not a quadratic polynomial, so $b^2 - 4ac$ is only the determinant. $\boxed{\text{A}}$

5) By inspection, $\frac{1}{9} + \frac{1}{16} + \frac{1}{25} << 1$, so $\frac{\hat{\mathbf{i}}}{3} + \frac{\hat{\mathbf{j}}}{4} + \frac{\hat{\mathbf{k}}}{5}$ is not a unit vector. For B), any hat vector is by definition a unit vector. For C), this is a one-dimensional unit vector in the negative $x$ direction. For D), $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} = 1$. $\boxed{\text{A}}$

6) The determinant is equal to $(x^2 + 8y) - (-y^2 - 6x) = x^2 + 6x + y^2 + 8y = 11$. Completing the square, this is equivalent to $(x+3)^2 + (y+4)^2 = 36$, a circle centered at $(-3, -4)$ having area $36\pi$. $\boxed{\text{E}}$

7) To find the new area, multiply the original area by the determinant of the mapping. $6\begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} = 6(18 - 10) = 6 \cdot 8 = 48.$ $\boxed{\text{A}}$

8) Multiplying one row of a matrix by a constant $k$ multiplies the determinant of the matrix by $k$ as well. All 5 rows of the matrix are multiplied by 2, so the determinant is multiplied by $2^5 = 32$, so $|2A| = 32\sqrt[5]{2}$. $\boxed{\text{D}}$

9) From the bottom two rows of the denominator and the left column of the numerator, we see that two of the equations are $-2x + 4y + 3z = -8$ and $5x - y = 7$. Adding these together yields $3x + 3y + 3z = -1$, or $x + y + z = -\frac{1}{3}$. $\boxed{\text{B}}$

10) This is equivalent for asking for the external angle of the angle opposite the side of length 7 in a $3 - 5 - 7$ triangle. By the Law of Cosines, $\cos\theta = \frac{9+25-49}{2 \cdot 3 \cdot 5} = -\frac{1}{2}$, so $\theta = 120°$ and the external angle is $60°$. $\boxed{\text{B}}$

11) The area of this parallelogram is the magnitude of the cross products of the vectors that represent adjacent sides. $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 4 & 0 \\ 5 & 1 & 0 \end{vmatrix} = -18\hat{\mathbf{k}}$, which has magnitude 18. $\boxed{\text{D}}$

12) By inspection, $A^2 = \begin{bmatrix} 4-9 & 18-36 \\ -2+4 & -9+16 \end{bmatrix} = \begin{bmatrix} -5 & -18 \\ 2 & 7 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 10-18 & 45-72 \\ -4+7 & -18+28 \end{bmatrix} = \begin{bmatrix} -8 & -27 \\ 3 & 10 \end{bmatrix}$. It can be shown through induction that $A^n = \begin{bmatrix} 1-3n & -9n \\ n & 3n+1 \end{bmatrix}$, which has a negative sum of entries of $8n-2$. Plugging in $n = 10$ gives a value of 78. $\boxed{\text{B}}$

13) The sum of the eigenvalues of a matrix is the trace of the matrix, which here is $2+11+23 = 36$. $\boxed{\text{C}}$

14) $\sum_{n=0}^{\infty} k^{-n} = \frac{1}{1-1/k} = \frac{k}{k-1}$. Thus, the sum is $\begin{bmatrix} 2 & \frac{3}{2} \\ \frac{4}{3} & \frac{5}{4} \end{bmatrix}$, whose sum of entries is $\frac{73}{12}$. $73+12 = 85$. $\boxed{\text{B}}$

15) Taking the dot product of the first with each of the other two, we obtain the system $(3a+9b)+(2-4a)+(4b-16) = 0$ and $(6a-3b)+(3-6a) = 0$. Simplifying, $-a+13b = 14$ and $-3b = -3$, giving $b = 1$ and $a = -1$. $a+b = 0$. $\boxed{\text{B}}$

16) This is equivalent to the scalar triple product via commutativity of the dot product and the ability to cycle vectors in the scalar triple product identity. Thus, the value is still equal to 2022. $\boxed{\text{D}}$

17) A notable counterexample to multiplicative commutativity in vector spaces is found in the vector space of square matrices of a fixed dimension. $\boxed{\text{C}}$

18) By inspection, the relevant value of $t$ is $t = 1$. The direction vectors of the lines when they intersect at $(5,-1,3)$ are $\langle 1,2,-2 \rangle$ and $\langle 8,-4,1 \rangle$. The dot product of these vectors is $8-8-2 = -2$, so the cosine of their intersections is $-\frac{2}{\sqrt{9}\sqrt{81}} = -\frac{2}{27}$. The cosine of the smaller angle is the negative of this, or $\frac{2}{27}$. $\boxed{\text{A}}$

19) $\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} = 2 \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} = 2 \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix}$. Thus, $(a_{n+1}, b_{n+1})$ is formed by rotating $(a_n, b_n)$ counterclockwise about the origin and dilating it about the origin by a factor of 2. From $(a_1, b_1)$ to $(a_{100}, b_{100})$, this process occurs 99 times. A rotation of $99 \cdot 30^\circ$ is equivalent to a rotation of $90^\circ$, meaning $(a_1, b_1)$ is on the line passing through the origin and $(4,-2)$. Scaling this point down by $2^{99}$ yields $(a_1, b_1) = \left(\frac{1}{2^{97}}, -\frac{1}{2^{98}}\right)$. $\frac{1}{2^{97}} - \frac{1}{2^{98}} = \frac{1}{2^{98}}$. $\boxed{\text{C}}$

20) Note that $\frac{1+i\sqrt{3}}{2} = e^{i\pi/3}$ and $\frac{1-i}{\sqrt{2}} = e^{-i\pi/4}$, so $\begin{bmatrix} \frac{1+i\sqrt{3}}{2} & 0 \\ 0 & \frac{1-i}{\sqrt{2}} \end{bmatrix}^n = \begin{bmatrix} e^{i\pi n/3} & 0 \\ 0 & e^{-i\pi n/4} \end{bmatrix}$. We are searching for the smallest positive value of $n$ such that both exponents are multiples of $2\pi$, which occurs at $LCM(6,8) = 24$. $\boxed{\text{D}}$

21) Carlos's vector for how far he travels in one second is perpendicular to the current. It is one leg of a right triangle with hypotenuse 10 (Carlos's speed in still water) and leg 5 (the current). Thus, Carlos crosses $5\sqrt{3}$ meters of river per second. $\frac{60}{5\sqrt{3}} = 4\sqrt{3}$. $\boxed{\text{B}}$

22) Since $||\hat{\mathbf{a}} \times \hat{\mathbf{b}}|| = ||\hat{\mathbf{a}}||||\hat{\mathbf{b}}|| \sin\theta$, $||\hat{\mathbf{a}} \times \hat{\mathbf{b}}||^2 = ||\hat{\mathbf{a}}||^2||\hat{\mathbf{b}}||^2(1-\cos^2\theta) = ||\hat{\mathbf{a}}||^2||\hat{\mathbf{b}}||^2 - (||\hat{\mathbf{a}}||||\hat{\mathbf{b}}||\cos\theta)^2 = ||\hat{\mathbf{a}}||^2||\hat{\mathbf{b}}||^2 - (\hat{\mathbf{a}} \cdot \hat{\mathbf{b}})^2$. Plugging in values, $||\hat{\mathbf{a}} \times \hat{\mathbf{b}}||^2 = 4^2 \cdot 5^2 - 15^2 = 400 - 225 = 175$ and $||\hat{\mathbf{a}} \times \hat{\mathbf{b}}|| = 5\sqrt{7}$. $\boxed{\text{D}}$

23) The vectors connecting $(1,1,1)$ to the other two points are $\langle 0,6,9\rangle$ and $\langle 4,2,0\rangle$. A vector normal to both of these can be found by taking the cross product $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 6 & 9 \\ 4 & 2 & 0 \end{vmatrix} = -18\hat{\mathbf{i}} + 36\hat{\mathbf{j}} - 24\hat{\mathbf{k}}$. Scaling this down gives the vector $\langle -3, 6, -4\rangle$ as perpendicular to both vectors, so the equation of the plane is $-3x + 6y - 4z = k$ for some value of $k$. Plugging in $(1,1,1)$ gives $k = -1$, so the equation of the plane is $3x - 6y + 4z = 1$. Plugging in $(13, 12, z)$ gives $39 - 72 + 4z = 1$, or $z = \frac{17}{2}$. $\boxed{\text{C}}$

24) The volume of a tetrahedron is one sixth the value of the scalar triple product of the vectors that represent its sides. Starting from $(0,4,0)$, these vectors are $\langle 4,-1,1\rangle$, $\langle 1,-4,1\rangle$, and $\langle 4,-2,0\rangle$. $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 4 & -1 & 1 \\ 1 & -4 & 1 \end{vmatrix} = \langle 3,-3,-15\rangle$, and $\langle 3,-3,-15\rangle \cdot \langle 4,-2,0\rangle = 12 + 6 + 0 = 18$. Thus, the volume of the tetrahedron is 3. $\boxed{\text{A}}$

25) The sum of the entries of the product of two matrices can be found in the following way. Construct a row vector where each entry is the sum of the entries in the corresponding column of the first matrix. Similarly, construct a column vector where each entry is the sum of the entries in the corresponding row of the second matrix. Multiply these together to obtain the desired sum. In the case of these matrices (which, notably, are digits of $\pi$ with some modifications made to make the calculations easier), we have the following.

$$\begin{bmatrix} 30 & 20 & 30 & 20 & 20 & 30 \end{bmatrix} \begin{bmatrix} 30 \\ 30 \\ 30 \\ 40 \\ 30 \\ 30 \end{bmatrix} = 900 + 600 + 900 + 800 + 600 + 900 = 4700$$

In case you were wondering, the actual matrix product is as follows.

$$\begin{bmatrix} 122 & 162 & 74 & 102 & 137 & 103 \\ 203 & 178 & 131 & 112 & 198 & 198 \\ 178 & 153 & 81 & 108 & 152 & 138 \\ 202 & 151 & 102 & 135 & 152 & 168 \\ 77 & 137 & 134 & 106 & 126 & 100 \\ 98 & 109 & 98 & 67 & 105 & 103 \end{bmatrix}$$

The sum of all of these entries, again, is 4700. $\boxed{\text{C}}$

26) If a matrix is singular, its determinant is 0. If a matrix is idempotent, its determinant must be 0 or 1. Both of these are made impossible by 29), so 26) is either B) or D). A 2-by-2 matrix in row-echelon form must either be upper triangular or have a determinant of 0 (the latter of which, once again, is made impossible by 29)). Because there is a unique answer to this question, the matrix must not be any of these and is instead lower triangular. $\boxed{\text{D}}$

27) Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. From 26), we know $b = 0$. Thus, $|M| = ad$. By 29), $ad \in \{12, 18, 24, 36\}$, and by 30), $a + d \in \{11, 13, 15, 19\}$. By examination, the possible pairs for $(a, d)$ are $(2, 9)$, $(3, 8)$, $(1, 12)$, $(4, 9)$, $(3, 12)$, and $(1, 18)$, where each pair is unordered (for now). Consider 27). If both C) and D) are true, then $M = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, which is impossible. Thus, $a < d$ and those pairs are now ordered. If the row sums are equal, then $c + d = a$ and the sum of the entries is $2a$, which has a maximum of 8, breaking 28). Thus, the column sums are equal, but the row sums are not. $\boxed{\text{C}}$

28) Because the column sums are equal and $b = 0$, the sum of the entries of $M$ is $2d$. Observing the ordered pairs, the possible values of this are 16, 18, 24, and 36. Only 16 is an answer choice. $\boxed{\text{B}}$

29) $d = 8$ is associated with $a = 3$. $|M| = ad = 24$. $\boxed{\text{C}}$

30) The trace of $M$ is $a + d = 11$. $\boxed{\text{A}}$

$$M = \begin{bmatrix} 3 & 0 \\ 5 & 8 \end{bmatrix}$$