MU 2021 Log1 Contest Answers

1) **(C)** Average value = $\frac{1}{0-(-4)}\int_{-4}^{0}\sin\frac{x}{2}\,dx = \frac{\cos 2 - 1}{2}$.

2) **(C)** To be divisible by 21, the number has to be divisible by 3 and 7; thus $3+7+3+0+N+5=18+N$ has to be a multiple of 3. That means N has to be 0, 3, 6, or 9. It's easy to just check each of these one by one to see if the original number is divisible by 7. It follows that 6 is the only one that works. The sum of the positive integral factors of 6 is $1+2+3+6=$ 12.

3) **(B)** Area = $\int_0^2 e^{-x}\,dx = -e^{-x}\Big]_0^2 = -e^{-2}+1$.

4) **(B)** Use your favorite technique of solving 3 x 3 systems to get $(x,y,z)=(4,5,1)$; thus $x^2+y^2+z^2 = 16+25+1=42$.

5) **(E)** Statement 1 is false in general; for example, take $f(x)=|x|$, which is continuous everywhere but is not differentiable at the origin. Statement 2 is true; it's actually a theorem. Statement 3 is also true because $f(x)\geq 3$ implies $\int_4^8 f(x)\,dx \geq \int_4^8 3\,dx = 12 > 9$.

6) **D**, suppose that we call the roots of $P(x)$ x, y, and z. The product of the roots taken two at a time is equal to $(xy)(yz)(xz) = (xyz)^2$. Thus, we are simply trying to find the product of the roots, squared. This can be found by Vieta's formula. The product of the roots is = $\frac{-constant}{a}$ if the degree of the highest term is odd and $\frac{constant}{a}$ if the degree of the highest term is even. In our case,

Product = $\frac{-(-16)}{1} = 16$

$16^2 = 256$

7) **C**,

I is true because if we have 2 corresponding angles then the third must also be a corresponding angle. With all three congruent angles we can determine that one triangle is simply a dilation of another as the sides are constructed from the angles.

II is false because if we have 2 corresponding sides we must have the included angle (the angle that lies at the vertex of the two sides) as opposed to any two angles. The included angle with the two sides proves congruency

III is true because if we have 3 corresponding and proportional sides between the triangles we know

(like stated in I) that the triangles are simply a dilation of each other.

Thus, only I and III are true.

8) **(E)** If $X$ is the number of heads, then $P(X=i)=\frac{C(7,i)}{2^7}$, where $C(n,i)=\frac{n!}{(n-i)!i!}$. Thus,

$$P(X\le 5 \mid X\ge 2)=\frac{P(2\le X\le 5)}{P(X\ge 2)}=\frac{(C(7,2)+C(7,3)+C(7,4)+C(7,5))/2^7}{(C(7,2)+C(7,3)+C(7,4)+C(7,5)+C(7,6)+C(7,7))/2^7}=\frac{14}{15}.$$

9) **(D)** The odd terms form a geometric series with first term of $\frac{1}{2}$ and common ratio of $\frac{1}{2}$; its sum is $\frac{1/2}{1-1/2}=1$. The even terms form a geometric series whose first term is $\frac{2}{3}$ and common ratio $\frac{1}{2}$; the sum of that is $\frac{2/3}{1-1/2}=\frac{4}{3}$. Altogether the sum is $1+\frac{4}{3}=\frac{7}{3}$.

10) **(D)** Let *ABCD* be a rectangle with $AB = 48$ and $BC = 55$. Fold the rectangle such that A coincides with *C*. Unfold the rectangle. Let *EF* be the crease, where *E* and *F* are on *AD* and *BC*, respectively. By symmetry, $BF = ED = x$. Now fold the rectangle again. Notice that the points *A* (which now coincides with *C*), *E*, and *D* make a right triangle with legs $x$ and 48 with hypotenuse $55 - x$. By the Pythagorean theorem, $(55-x)^2 = x^2 + 48^2$; this yields $x = 721/110$. Now unfold the rectangle again. Notice the length of the crease *L* is the hypotenuse of a triangle with legs of 48 and $55 - 2x$. Thus, $L = \sqrt{48^2 + (55-2x)^2} = \sqrt{48^2 + (55-2(\frac{721}{110}))^2} = \dfrac{3504}{55}$, making $m + n = 3559$.

11) **A** Factoring this expression yields $\left(\tan(x)+\dfrac{\sqrt{3}}{3}\right)\left(\tan x+\sqrt{3}\right)$, which has zeroes where tangent is the opposite of the square root of three, or the opposite of its reciprocal. The sum of these angles on the requested interval are

$$\frac{-\pi}{6}+\frac{-\pi}{3}+\frac{2\pi}{3}+\frac{5\pi}{6}=\pi$$

12) **(D)** Differentiating implicitly, we get that $\dfrac{dy}{dx} = \dfrac{3x^2y - y^2}{2xy - x^3}$; the curve will have a horizontal tangent line when this quantity is 0. This occurs when $3x^2y - y^2 = y(3x^2 - y) = 0$. If $y = 0$, then we have $x(0)^2 - x^3(0) = 0 = 6$, a contradiction. Thus, $3x^2 = y$; plug this into the original equation to get $9x^5 - 3x^5 = 6$, or $x = 1$, making $y = 3$. The sum of the coordinates is 4.

13) **B** This infinite sum is geometric. Observe that by change of base formula, we can phrase each term in log base 3, because $\log_9(x) = \dfrac{\log_3(x)}{\log_3(9)}$ and $\log_{81}(x) = \dfrac{\log_3(x)}{\log_3(81)}$, so there is a common ratio of ½. By infinite sum of a geometric series, the sum is $2\log_3(x)$, which has a solution when $x = 3^9$.

14) This is a Caesar cipher with a shift of 13, also known as ROT13.

**Never gonna give you up, never gonna let you down, never gonna run around and desert you.**

15) This is a keyword cipher using the keyword ''PRASEODYMIUM." A keyword cipher works similarly to a cryptogram except that a keyword appears at the leftmost part of the decryption table, as follows.

Ciphertext: ABCDEFGHIJKLMNOPQRSTUVWXYZ

Plaintext: PRASEODYMIUBCFGHJKLNQTVWXZ

**I am serious, and don't call me Shirley.**