Mu Bowl Answers and Solutions

ANSWERS :

Question	A	B	C	D	Combined
Q0	0				**0**
Q1	$\frac{7}{8}$	$4e$	$-\frac{\pi^2}{6}$	$\frac{4-2\pi}{\pi^2}$	$\boldsymbol{7\pi - 14}$ **or** $\boldsymbol{-14 + 7\pi}$
Q2	2	$\frac{7}{6}$	$\frac{128}{105}$	$\frac{1}{3}$	$\boldsymbol{\frac{128}{135}}$
Q3	$\frac{1}{2}$	4	$\frac{1}{2e}$	1	$\boldsymbol{\frac{1}{e}}$
Q4	-4	8π	12	1	$\boldsymbol{-384\pi}$
Q5	$\frac{32}{3}$	$\frac{2048\pi}{15}$	8π	$\frac{32}{5}$	$\boldsymbol{\frac{256}{25}}$
Q6	-1	$\frac{1}{\sqrt{3}}$	$\frac{4}{\sqrt{17}}$	1	$\boldsymbol{\frac{16}{51}}$
Q7	$\frac{1}{32}$	4π			$\boldsymbol{\frac{\pi}{8}}$
Q8				$12\sqrt{2}\sqrt{2+\sqrt{2}}$	**24**
Q9	$\frac{1}{2}$	$1 - 2\ln 2$			**4**
Q10	1	4	2	e	$\boldsymbol{8e}$
Q11					**3**
Q12	e^2	$2e$	$-\frac{\pi}{2}$	1011	$\boldsymbol{-1011\pi e^2}$
Q13	-704	0	0		$\boldsymbol{-704}$
Q14					**2020**

SOLUTIONS :

0. By Vieta's the sum of the solutions is 0.

1.

A. $\frac{x^5+1}{x^4} = x + \frac{1}{x^4}$. Use the power rule and plug the value in to get $\frac{7}{8}$.

B. $\frac{d}{dx}\left(x^2\, ln^2\, x\right) = 2x\ln^2 x + 2x\ln x$. Plug in to get $4e$.

C. Use $arcsin\, x + arccos\, x = \frac{\pi}{2}$ to get to $\frac{d}{dx}\left(\frac{\pi}{2}\, arcsin\, x\; arccos\, x\right) = \frac{-\pi^2}{6}$ after applying the product rule and plugging in.

D. Use the quotient rule and FTC II to get the derivative is equal to $\frac{x^2 cosx - \int_0^x tcost\, dt}{x^2}$. At $x = \frac{\pi}{2}$, this is equal to $\frac{4-2\pi}{\pi^2}$.

$\frac{6ABCD}{e} = 7\pi - 14$ or $-14 + 7\pi$

2.

A. Using integration by parts, we get that $\int_0^1 ln^2\, x\, dx = x\, ln^2\, x - 2x\, ln\, x + 2x|^1{}_0 = 2$

B. $\int_0^{\frac{\pi}{4}} \cos 2x\ (sin\, x + cos\, x)^4\, dx = \int_0^{\frac{\pi}{4}} \cos 2x\, (1 + \sin 2x)^2\, dx$.Using $u = 1 + sin\, 2x$, we get that $\int_0^{\frac{\pi}{4}} \cos 2x\, (1 + \sin 2x)^2\, dx = \frac{7}{6}$.

C. $\int_0^2 x^2(2-x)^4\, dx = \int_0^2 x^4(2-x)^2 dx = \int_0^2 x^6 - 4x^5 + 4x^4\, dx = \frac{128}{105}$

D. $\int_{-1}^{1} \frac{x^2}{e^x+1} dx = \int_{-1}^{1} \frac{x^2}{e^{-x}+1}\, dx = \int_{-1}^{1} \frac{x^2 e^x}{e^x+1} dx$. Noting that these two integrals are equal and adding them together yields that the value is equal to $\frac{1}{2}\int_{-1}^{1} x^2 dx = \frac{1}{3}$.

$$ABCD = \frac{128}{135}.$$

3.

A. By L'Hospital 3 times or Taylor Expansion, $\frac{1}{2}$.

B. Asymptotically, this becomes $(x+2) - (x-2) = 4$.

C. $\lim_{n\to\infty} \frac{n}{2}\left(\frac{1}{n}\right)^2\left(1-\frac{1}{n}\right)^{n-2} = \lim_{n\to\infty} \frac{n}{2}\left(\frac{1}{n}\right)^2\left(\frac{n}{n-1}\right)^2\left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty} \frac{n}{2}\left(\frac{1}{n-1}\right)^2\left(1-\frac{1}{n}\right)^n = \frac{1}{2e.}$

D. When going from the positive side, $f(arctan\, x) = arctan\, x$, so this limit is equal to 1.

$$ABCD = \frac{1}{e}.$$

4.

A. The function approaches $(-1)^4 - 4 - 1$ from both sides, so the limit is -4.
B. By symmetry, this is equal to $\int_{-\pi}^{\pi} 4x\ sin\ x\ dx = 8\pi$.
C. The global minimum would occur at $x = -1$, but this point is replaced by a point that is large, so there is no minimum. 12.
D. By Descartes' Rule of Signs, there is exactly 1 positive real root.

$$ABCD = -384\pi$$

5.

A. $\int_{-2}^{2}(4 - x^2)\ dx = \frac{32}{3}$.
B. $\pi \int_{-2}^{2} 8^2 - (x^2 + 4)^2\ dx\ =\ 2\pi \int_0^2 48 - 8x^2 - x^4\ dx = \frac{2048\pi}{15}$.
C. $2\pi \int_0^2 x(4 - x^2)\ dx\ =\ 8\pi$
D. Using Pappus' Theorem, the answer is $\frac{32}{5}$.

$$\frac{BD}{AC} = \frac{256}{25}$$

6.

A. $f(-1) = -3$ and $f(0) = 2$. Because f is continuous, then the IVT guarantees a root between these values. $a = -1$.
B. The guaranteed value is a value of c such that $f'(c) = \frac{7-2}{1-0} = 5$. The value that occurs in this interval is $\frac{1}{\sqrt{3}}$.
C. $f'(0) = 4$, so the line normal will have slope $-\frac{1}{4}$. This will make an angle of $arctan\ 4$ with the positive y axis, so $sin\ (arctan\ 4) = \frac{4}{\sqrt{17}}$
D. The tangent line at $x = 0$ is $y = 4x + 2$. $x^3 + 4x + 2 = 4x + 2\ \rightarrow x^3 = 0$ has only one solution, so the line intersects 1 time.

$$A^2B^2C^2D^2 = \frac{16}{51}$$

7.

A. $\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = \frac{-2t}{(1+t^2)^2}$. $\frac{dy}{dx} = -\frac{t}{2(1+t^2)^2}$. $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{3t^2-1}{8(1+t^2)^3}$. At $t = 1$, this is $\frac{1}{32}$.

B. We can compute the area through $\int_{-\infty}^{\infty} y dx = \int_{-\infty}^{\infty} \frac{4}{1+t^2}\ dt = 4\pi$.

$$AB = \frac{\pi}{8}.$$

8.

The bug will travel on the curve $r = 3 + 3\ cos\ \theta$ between 0 and $\frac{\pi}{4}$ and $\frac{5\pi}{4}$ and 2π, and on $r = 3 + 3\ sin\ \theta$ from $\frac{\pi}{4}$ to $\frac{5\pi}{4}$. The total distance traveled is then

$$\int_0^{\frac{\pi}{4}} \sqrt{(3 + 3\ cos\ \theta)^2 + (-3\ sin\ \theta)^2} d\theta + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sqrt{(3 + 3\ sin\ \theta)^2 + (3\ cos\ \theta)^2} d\theta$$
$$+ \int_{\frac{5\pi}{4}}^{2\pi} \sqrt{(3 + 3\ cos\ \theta)^2 + (-3\ sin\ \theta)^2} d\theta$$

You can use the Pythagorean Identity and then the sine half angle formula for the $cos\ \theta$ integrals and the substitution $u^2 = 1 + sin\ \theta, 2u\ du\ = cos\ \theta\ d\theta$ for the middle one. This will give you $d = 12\sqrt{2}\sqrt{2+\sqrt{2}}, d\sqrt{2-\sqrt{2}} = 24$.

9.

A. On each interval $(n, n+1)$ for n an integer, $\{x\}$ goes from $(0,1)$ and $[x] = n$. So we can write this integral as $\sum_{n=0}^{\infty} \int_0^1 \frac{x}{1+n}\ dx = \sum_{n=0}^{\infty} \frac{1}{2(1+n)}$. Thus, the limit is equal to $\frac{1}{2}$.

B. Consider each interval $\left(\frac{1}{n+1}, \frac{1}{n}\right)$ for n an integer. The integral becomes $\sum_{n=1}^{\infty} \int_0^1 \left(\frac{1}{n} - \frac{1}{n+1}\right)(-1)^n dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} - \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$. Using the sum of alternating harmonic series, this becomes $-\ln 2 - (\ln 2 - 1) = 1 - 2\ln 2$.

$$e^{2A-B} = 4$$

10.

A. $f'(x) = \frac{2-2x^2}{(x^2+1)^2}$. The maximum will occur at $x = 1$.

B,C. We can factor $g(x,y) = 6 + (x-2y)^2 + (x-4)^2 + (y-2)^2$. The minimum value occurs when all the squared terms are 0, so at $(x,y) = (4,2)$.

D. Using logarithmic differentiation, we can get $h'(x) = h(x)\left(\frac{1-lnx}{x^2}\right)$. This has a maximum value occurring when $x = e$.

$$ABCD = 8e$$

11.

I. This one is equal to e.

II. We can use the exponential of the logarithm of the limit and L'Hospital to find this limit DNE.

III. L'Hospital twice gives this limit is equal to $\frac{1}{2}$.

IV. Using either L'Hospital three times or Taylor polynomials, you can get that this limit is 4.

V. This limit does not exist as the numerator being nonzero causes the limit to blow up.

VI. This limit does not exist as the denominator grows faster than the numerator shrinks around $x = 3$.

Answer: 3

12.

A. By the Maclaurin expansion of e^x, this part is e^2.

B. $\sum_{n=0}^{\infty} \frac{n^2}{n!} = 0 + \sum_{n=1}^{\infty} \frac{n}{(n-1)!} = \sum_{n=0}^{\infty} \frac{n}{n!} + \frac{1}{n!} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} + e = 2e$.

C. This series will telescope and the remaining term will be $\lim_{n\to\infty} -\arctan(n) = -\frac{\pi}{2}$.

D. Use the substitution $u = 2021 - x$ to get the symmetric integral. Add these two sums together and observe that the general term becomes 1. The sum becomes $\frac{2022}{2} = 1011$.

$$ABCD = -1011\pi e^3$$

13.

A. Logarithmic differentiation gives $\frac{f'(x)}{f(x)} = \frac{6}{x-1} + \frac{8}{2x+1}$, so $f'(-1) = 64(-3-8) = -704$
B. $3/2$ is out of the radius of convergence of f, 0.
C. $\frac{df(\pi)}{dx}$ is taking the derivative of a constant. This is therefore just equal to 0.

$$A + B + C = -704$$

14.
Multiply both sides by $(x-1)$ to get $(x-1)f(x) = x^{2021}-1$. Take the derivative with respect to x to get $(x-1)f'(x)+f(x) = 2021x^{2020}$. Now, plugging in 2, we get $f'(2)+f(2) = 2021(2^{2020})$. Thus, the quantity requested is 2020.