1 Answers

1. C
2. E
3. A
4. C
5. C
6. C
7. B
8. A
9. D
10. C
11. D
12. A
13. B
14. B
15. C
16. C
17. A
18. B
19. A
20. A
21. D
22. D
23. C
24. D
25. B
26. A
27. B
28. D
29. A
30. B

2 Solutions

1. When written in closed form, the expression $1^{2022} + 2^{2022} + \cdots + n^{2022}$ will be a polynomial in n with leading term an^{2023} for some a. Find a.

   Solution. In particular, we must have, letting $P(n)$ be the closed form polynomial, that

$$1 = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} k^{2022}}{P(n)} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} k^{2022}}{an^{2023}}$$

   Hence,

$$a = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^{2022} \frac{1}{n} = \int_0^1 x^{2022} dx = \frac{1}{2023},$$

   and the answer is (C).

2. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x}{1+5x^4}\, dx$.

   Solution. This is a convergent integral as $x/(1+5x^4) < 1/(5x^3)$ and $\int_0^\infty 1/(5x^3)\, dx$ converges. Hence, as the integrand is odd and the bounds are symmetric, the value is 0. (E)

3. Evaluate $\displaystyle\int_0^1 \frac{x^2}{x^6+1}\, dx$.

   Solution. Let $u = x^3$ so that $du = 3x^2\, dx$. The integral becomes

$$\frac{1}{3}\int_0^1 \frac{1}{1+u^2}\, du = \frac{1}{3}\arctan u\Big|_0^1 = \frac{\pi}{12}.$$

   So the answer is (A).

4. Evaluate $\displaystyle\lim_{n\to\infty} n\int_1^{2022} \frac{1}{1+x^n}\, dx$.

   Solution. Let $u = x^{-1}$. Then $x = u^{-1}$ and $dx = -u^{-2}\, du$. So this u-sub gives

$$\begin{aligned}\lim_{n\to\infty} n\int_{1/2022}^{1} \frac{1}{1+u^{-n}}u^{-2}\, du &= \lim_{n\to\infty} n\int_{1/2022}^{1} \sum_{k=1}^{\infty}(-1)^{k+1}u^{kn-2}\, du\\ &= \sum_{k=1}^{\infty}(-1)^{k+1}\lim_{n\to\infty}\int_{1/2022}^{1} nu^{kn-2}\, du\\ &= \sum_{k=1}^{\infty}(-1)^{k+1}\lim_{n\to\infty}\frac{n}{kn-1}\left(1-\frac{1}{2022^{kn-1}}\right)\\ &= \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = \ln(2),\end{aligned}$$

   where we have used the fact that this converges absolutely to switch order of summation, integral, and limit. So the answer is (C).

5. Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{n!x^n}{n^n}$.

Solution. By Stirling's approximation, $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, so the sum is asymptotic to $\sum_{n=1}^{\infty} \frac{x^n\sqrt{2\pi n}}{e^n}$, which has a radius of convergence of e. (C)

6. A solution to the differential equation $\frac{dy}{dx} = 3x^2y + 9x^2 + y + 3$ passes through the origin and $(1, k)$. Find k.

Solution. We have $\frac{dy}{dx} = (3x^2 + 1)(y + 3)$. Separating, $\frac{dy}{y+3} = (3x^2 + 1)\ dx$. Integrating, $\ln|y + 3| = x^3 + x + C$, so $y = Ce^{x^3+x} - 3$. Setting $x = y = 0$ gives $C = 3$, so $y = 3e^{x^3+x} - 3$ and $y(1) = 3e^2 - 3$. (C)

7. What is the area of the region bounded by $r = 4 + 3\cos\theta$ in the polar plane?

Solution. The area is given by

$$\begin{aligned}\frac{1}{2}\int_0^{2\pi}(4 + 3\cos\theta)^2\ d\theta &= \frac{1}{2}\int_0^{2\pi}\left(16 + 24\cos\theta + 9\cos^2\theta\right)\ d\theta \\ &= \frac{1}{2}\int_0^{2\pi}\left(\frac{41}{2} + 24\cos\theta + \frac{9}{2}\cos 2\theta\right)\ d\theta = \frac{41\pi}{2}\end{aligned}$$

since we integrate through complete periods of cosine. (B)

8. Find the slope of the line tangent to $x^2y + 2xy^3 - x^4 = 2$ at the point $(1, 1)$.

Solution. Deriving implicitly, $2xy + x^2\frac{dy}{dx} + 2y^3 + 6xy^2 - 4x^3 = 0$, so $\frac{dy}{dx} = -\frac{2xy+2y^3-4x^3}{x^2+6xy^2}$. Plugging in $x = y = 1$ gives $\frac{dy}{dx} = 0$. (A)

9. Evaluate $\lim_{x\to 0} \frac{\sin(x^2\sin(x^2)) + x\sin(\sin(2x))}{\sin(2x\sin(x^2)) + \sin(\sin(\sin(x\sin(x))))}$.

Solution. Using that $\sin(x) = x + O(x^3)$ near $x = 0$ can easily give us that the numerator is $(x^4 + O(x^8)) + (2x^2 + O(x^4)) \to 2x^2$ as $x \to 0$ and the denominator is $(2x^3 + O(x^7)) + (x^2 + O(x^4)) \to x^2$ as $x \to 0$. Therefore the ratio is 2 as $x \to 0$. (D)

10. Find the volume of the solid that results when the area between the curve $y = e^{2x}$ and the lines $y = 0$, $x = 1$, and $x = 2$ is rotated around the x-axis.

Solution. $\pi\int_1^2((e^{2x})^2 - 0^2)\ dx = \pi\int_1^2 e^{4x}\ dx = \frac{\pi(e^8-e^4)}{4}$. (C)

11. Evaluate $\int_1^{2021}(x - 1)(x - 2)\cdots(x - 2021)\ dx$.

Solution. Let $u = x - 1011$. Then we get

$$\begin{aligned}\int_1^{2021}(x - 1)(x - 2)\cdots(x - 2021)\ dx &= \int_{-1010}^{1010}(u + 1010)(u + 1009)\cdots(u - 1010)\ du \\ &= \int_{-1010}^{1010}(u^2 - 1010^2)(u^2 - 1009^2)\cdots(u^2 - 1)u\ du.\end{aligned}$$

The integrand is odd and the bounds are symmetric, so the integral has value 0. (D)

12. Determine the convergence or divergence of the infinite series

$$\frac{1}{\ln(4)\ln(2)}+\frac{1}{\ln(27)\ln(3)}+\frac{1}{\ln(256)\ln(4)}+\frac{1}{\ln(3125)\ln(5)}+\frac{1}{\ln(46656)\ln(6)}+\cdots.$$

Solution. The general term of the series is

$$\frac{1}{\ln(n^n)\ln(n)}=\frac{1}{n\ln^2(n)}$$

starting with $n=2$. By the integral test,

$$\int_2^\infty \frac{1}{x\ln^2(x)}\,dx=\lim_{b\to\infty}\left.\frac{-1}{\ln(x)}\right|_2^b=\lim_{b\to\infty}\left(-\frac{1}{\ln(b)}+\frac{1}{\ln(2)}\right)=\frac{1}{\ln(2)}.$$

Hence, the series is absolutely convergent. (A)

13. Evaluate $\displaystyle\int_1^2 \frac{3x^2-4}{x^3-4x+5}\,dx.$

Solution. The numerator is the derivative of the denominator, so $u=x^3-4x+5$ gives $\int_2^5 \ln u\,du=\ln\left(\frac{5}{2}\right)$. (B)

14. Find the length of the polar curve $r=\sqrt{1+\cos(2\theta)}$ over the interval $[0,2\pi]$.

Solution. The arc length of a polar curve is given by

$$\int_\alpha^\beta \sqrt{(r')^2+r^2}\,d\theta.$$

We have $r=\sqrt{1+2\cos(2\theta)}$ so $r'=-\frac{1}{2}(1+\cos(2\theta))^{-1/2}(2\sin(2\theta))=\dfrac{\sin(2\theta)}{\sqrt{1+\cos(2\theta)}}$. Thus, we compute

$$\int_0^{2\pi}\sqrt{\frac{\sin^2(2\theta)}{1+\cos(2\theta)}+1+\cos(2\theta)}\,d\theta=\int_0^{2\pi}\sqrt{2}\,d\theta=2\pi\sqrt{2}.$$

So the answer is (B).

15. Compute $\displaystyle\int_0^1 \arctan\sqrt{x}\,dx.$

Solution. Let $u^2=x$, so $2u\,du=dx$. The integral then becomes $\int_0^1 2u\arctan u\,du$. Using integration by parts to get the antiderivative results in

$$\begin{aligned}u^2\arctan u-\int\frac{u^2}{1+u^2}\,du&=u^2\arctan u-\int\left(1-\frac{1}{1+u^2}\right)\,du\\&=u^2\arctan u-u+\arctan u+C\\&=(u^2+1)\arctan u-u+C\\&=(x+1)\arctan\sqrt{x}-\sqrt{x}+C.\end{aligned}$$

Evaluating from 0 to 1, we get $2\arctan 1-1=\pi/2-1=(\pi-2)/2$. (C)

16. How many continuous functions f with a domain of $[0,1]$ satisfy this integral equation? $\int_0^1 (f(x))^2\,dx = \int_0^1 (f(x))^3\,dx = \int_0^1 (f(x))^4\,dx$

Solution. We then have

$$0 = \int_0^1 f(x)^2\,dx - 2\int_0^1 f(x)^3\,dx + \int_0^1 f(x)^4\,dx = \int_0^1 f(x)^2(f(x)-1)^2\,dx$$

As f is continuous and the integrand is always nonnegative, it follows that on $[0,1]$ we have $f(x)^2(f(x)-1)^2 = 0$, so for each $x \in [0,1]$ either $f(x) = 0$ or $f(x) = 1$. As f is continuous it must be that $f \equiv 0$ or $f \equiv 1$, giving the answer 2. (C)

17. Find $\frac{d^2y}{dx^2}$ where $x = t^2$ and $y = t^2 + t$.

Solution. We obtain $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+1}{2t} = 1 + \frac{1}{2t}$ and so

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{\frac{d}{dt}\left(1+\frac{1}{2t}\right)}{\frac{dx}{dt}} = \frac{-1/(2t^2)}{2t} = -\frac{1}{4t^3}.$$

So the answer is (A).

18. A value of θ is uniformly randomly selected from the range $\left[\frac{\pi}{6}, \frac{\pi}{4}\right]$. Find the expected value of $\sec^2\theta$.

Solution. Note that $\frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$. We are looking for $\frac{12}{\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2\theta\,d\theta = \frac{12}{\pi}\left(\tan\frac{\pi}{4} - \tan\frac{\pi}{6}\right) = \frac{12-4\sqrt{3}}{\pi}$. (B)

19. Find $f^{(5)}(0)$, where $f(x) = \arctan(x)$.

Solution. As $f'(x) = 1/(1+x^2) = 1 - x^2 + x^4 - \cdots$ we seek the fourth derivative, which is just 4! times the coefficient of x^4 of the Maclaurin series, which is 1. Hence the answer is $4! = 24$. (A)

20. Let $f(x) = x^{\ln(x)}$. Evaluate $f'(2)$.

Solution. Taking the natural logarithm of both sides and differentiating gives

$$\begin{aligned}\ln(f(x)) &= \ln(x^{\ln(x)}) = (\ln(x))^2\\ \frac{f'(x)}{f(x)} &= \frac{2\ln(x)}{x}\\ f'(x) &= \frac{2f(x)\ln(x)}{x} = \frac{2x^{\ln(x)}\ln(x)}{x}.\end{aligned}$$

Hence $f'(2) = \frac{2\cdot 2^{\ln(2)}\ln(2)}{2} = 2^{\ln(2)}\ln(2)$. (A)

21. A particle moving in the xy-plane has acceleration vector $\mathbf{a}(t) = (9t^2 - 4)\mathbf{i} + (4t+1)\mathbf{j}$ for all $t \geq 0$, and it has velocity vector $\mathbf{v}(t) = -\mathbf{i} - 2\mathbf{j}$ at time $t = 0$. What is the speed of the particle at time $t = 2$?

Solution. Integrating the vector $\mathbf{a}(t)$, we get

$$\mathbf{v}(t) = \int \mathbf{a}(t)\ dt = (3t^3 - 4t + C_1)\mathbf{i} + (2t^2 + t + C_2)\mathbf{j}.$$

Using the initial value, we find that $C_1 = -1$ and $C_2 = -2$ so that $\mathbf{v}(t) = (3t^3 - 4t - 1)\mathbf{i} + (2t^2 + t - 2)\mathbf{j}$. At time $t = 2$, we have $\mathbf{v}(2) = 15\mathbf{i} + 8\mathbf{j}$. The speed of the particle is therefore $\sqrt{15^2 + 8^2} = 17$. (D)

22. Compute $\displaystyle\int_0^\infty \frac{\lfloor x \rfloor}{(1+x)^2}\ dx$ where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Solution. We can rewrite this as

$$\sum_{n=0}^\infty \int_n^{n+1} \frac{n}{(1+x)^2}\ dx = \sum_{n=0}^\infty \frac{-n}{1+x}\bigg|_n^{n+1} = \sum_{n=0}^\infty \left(\frac{n}{n+1} - \frac{n}{n+2}\right) = \sum_{n=0}^\infty \frac{n}{(n+1)(n+2)}$$

which diverges by the limit comparison with the harmonic series. (D)

23. Let $f(x) = (x^2+3)^3$. Evaluate $\dfrac{dy}{d\sqrt{x}}$ at $x = 1$.

Solution. We have that

$$\frac{dy}{d\sqrt{x}} = \frac{dy}{dx}\frac{dx}{d\sqrt{x}} = \frac{dy}{dx}\frac{1}{\frac{d\sqrt{x}}{dx}} = 6x(x^2+3)^2\frac{1}{1/(2\sqrt{x})} = 12x(x^2+3)^2\sqrt{x}$$

Evaluated at 1, we obtain $12 \cdot 4^2 \cdot 1 = 192$. (C)

24. Find the y-intercept of the tangent line to the curve defined parametrically by $x = e^{3t} + 2$ and $y = \ln(e^{6t} + 4e^{3t} + 4)$ at the point where $t = \ln 2$.

Solution. Note that $y = \ln(e^{6t} + 4e^{3t} + 4) = \ln(e^{3t}+2)^2 = 2\ln(e^{3t}+2) = 2\ln(x)$. Hence, we can use the Cartesian equation to find $y'(x) = 2/x$. When $t = \ln 2$, then $x = e^{3\ln 2} + 2 = e^{\ln 8} + 2 = 10$ and $y = 2\ln 10$. It follows that the tangent line is $y = 2\ln 10 + (x - 10)/5$, and the y-intercept is $y(0) = 2\ln 10 - 2$. (D)

25. Evaluate $\displaystyle\lim_{x\to 1} \frac{(\sqrt[3]{x}-1)(\sqrt[4]{x}-1)(\sqrt[5]{x}-1)}{(x-1)^3}$.

Solution. Factoring $x - 1$ as a difference of cubes, fourth powers, and fifth powers yields

$$\begin{aligned}\frac{(\sqrt[3]{x}-1)(\sqrt[4]{x}-1)(\sqrt[5]{x}-1)}{(x-1)^3} &= \frac{(\sqrt[3]{x}-1)(\sqrt[4]{x}-1)(\sqrt[5]{x}-1)}{(x-1)(x-1)(x-1)} \\ &= \frac{1}{(x^{2/3}+x^{1/3}+1)(x^{3/4}+x^{2/4}+x^{1/4}+1)(x^{4/5}+x^{3/5}+x^{2/5}+x^{1/5}+1)}\end{aligned}$$

so the limit as $x \to 1$ is $1/(3 \cdot 4 \cdot 5) = 1/60$. (B)

26. Evaluate $\lim_{n\to\infty} \frac{1}{n!} \sum_{k=1}^{\infty} \frac{1}{\binom{n+k}{k} k!}$.

Solution. We have

$$\lim_{n\to\infty} \frac{1}{n!} \sum_{k=1}^{\infty} \frac{1}{\binom{n+k}{k} k!} = \lim_{n\to\infty} \frac{1}{n!} \sum_{k=1}^{\infty} \frac{1}{\frac{(n+k)!}{k!n!} k!} = \lim_{n\to\infty} \frac{1}{n!} \sum_{k=1}^{\infty} \frac{k!n!}{(n+k)!k!} = \lim_{n\to\infty} \frac{n!}{n!} \sum_{k=1}^{\infty} \frac{k!}{(n+k)!k!}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{(n+k)!} = \lim_{n\to\infty} \left(\sum_{k=0}^{\infty} \frac{1}{k!} - \sum_{k=0}^{n} \frac{1}{k!} \right) = e - e = 0$$

So the answer is (A).

27. Evaluate $\int_1^e \frac{x-1}{x+x^2\ln(x)}\, dx$.

Solution. Divide numerator and denominator by x^2 to get

$$\int_1^e \frac{x-1}{x+x^2\ln(x)}\, dx = \int_1^e \frac{1/x - 1/x^2}{1/x + \ln(x)}\, dx.$$

Letting $u = 1/x + \ln(x)$ we get $du = (-1/x^2 + 1/x)\, dx$ so that

$$\int_1^e \frac{1/x - 1/x^2}{1/x + \ln(x)}\, dx = \int_1^{1+1/e} \frac{1}{u}\, du = \ln\left(1 + \frac{1}{e}\right) = \ln\left(\frac{e+1}{e}\right) = \ln(e+1) - 1.$$

So the answer is (B).

28. A particle's movement in the coordinate plane is parametrized by $x = \sin^2\theta$ and $y = \cos(2\theta)$. Find the total distance (not displacement) the particle travels as t increases from 0 to 2022π.

Solution. Note that $\cos(2\theta) = 1 - 2\sin^2\theta$, so this is the line segment $y = 1 - 2x$ as x ranges from 0 to 1; it has length $\sqrt{5}$. The particle moves from $(0,1)$ to $(1,-1)$ in the range $t \in \left[0, \frac{\pi}{2}\right]$, and then back in $t \in \left[\frac{\pi}{2}, \pi\right]$, for a total distance of $2\sqrt{5}$. This process will repeat 2021 more times, so the total distance the particle moves in $t \in [0, 2022\pi]$ is $4044\sqrt{5}$. (D)

29. Everybody knows l'Hôpital's rule. But do you know the namesake mathematician's first name? (Hint: He is French.)

Solution. It is Guillaume. (A)

30. Evaluate $\int 2022x^{2021}\, dx$.

Solution. The antiderivative is $x^{2022} + C$. (B)