1 Answers

1. C
2. E
3. A
4. C
5. C
6. C
7. B
8. A
9. D
10. C
11. D
12. A
13. B
14. B
15. C
16. C
17. A
18. B
19. A
20. A
21. D
22. D
23. C
24. D
25. B
26. A
27. B
28. D
29. A
30. B
2 Solutions

1. When written in closed form, the expression $1^{2022} + 2^{2022} + \cdots + n^{2022}$ will be a polynomial in $n$ with leading term $a_n^{2023}$ for some $a$. Find $a$.

Solution. In particular, we must have, letting $P(n)$ be the closed form polynomial, that

$$1 = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{k^{2022}}{P(n)} = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{k^{2022}}{a_n^{2023}}$$

Hence,

$$a = \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{k}{n} \right)^{2022} = \int_{0}^{1} x^{2022} \, dx = \frac{1}{2023},$$

and the answer is (C).

2. Evaluate $\int_{-\infty}^{\infty} \frac{x}{1 + 5x^4} \, dx$.

Solution. This is a convergent integral as $x/(1+5x^4) < 1/(5x^3)$ and $\int_{0}^{\infty} 1/(5x^3) \, dx$ converges. Hence, as the integrand is odd and the bounds are symmetric, the value is 0. (E)

3. Evaluate $\int_{0}^{1} \frac{x^2}{x^6 + 1} \, dx$.

Solution. Let $u = x^3$ so that $du = 3x^2 \, dx$. The integral becomes

$$\frac{1}{3} \int_{0}^{1} \frac{1}{1 + u^2} \, du = \frac{1}{3} \arctan u \bigg|_{0}^{1} = \frac{\pi}{12}.$$ 

So the answer is (A).

4. Evaluate $\lim_{n \to \infty} n \int_{1}^{2022} \frac{1}{1 + x^n} \, dx$.

Solution. Let $u = x^{-1}$. Then $x = u^{-1}$ and $dx = -u^{-2} \, du$. So this $u$-sub gives

$$\lim_{n \to \infty} n \int_{1/2022}^{1} \frac{1}{1 + u^{-n}} \, du = \lim_{n \to \infty} n \int_{1/2022}^{1} \sum_{k=1}^{\infty} (-1)^{k+1} u^{kn-2} \, du$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} \lim_{n \to \infty} \int_{1/2022}^{1} n u^{kn-2} \, du$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} \lim_{n \to \infty} \frac{n}{kn-1} \left( 1 - \frac{1}{2022^{kn-1}} \right)$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = \ln(2),$$

where we have used the fact that this converges absolutely to switch order of summation, integral, and limit. So the answer is (C).
5. Find the radius of convergence of \( \sum_{n=1}^{\infty} \frac{n!x^n}{n^n} \).

**Solution.** By Stirling’s approximation, \( n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \), so the sum is asymptotic to \( \sum_{n=1}^{\infty} \frac{x^n}{e^n} \), which has a radius of convergence of \( e \). (C)

6. A solution to the differential equation \( \frac{dy}{dx} = 3x^2 y + 9x^2 + y + 3 \) passes through the origin and \((1, k)\). Find \( k \).

**Solution.** We have \( \frac{dy}{dx} = (3x^2 + 1)(y + 3) \). Separating, \( \frac{dy}{y+3} = (3x^2 + 1) \, dx \). Integrating, \( \ln |y + 3| = x^3 + x + C \), so \( y = Ce^{x^3 + x} - 3 \). Setting \( x = y = 0 \) gives \( C = 3 \), so \( y = 3e^{x^3 + x} - 3 \) and \( y(1) = 3e^2 - 3 \). (C)

7. What is the area of the region bounded by \( r = 4 + 3 \cos \theta \) in the polar plane?

**Solution.** The area is given by

\[
\frac{1}{2} \int_0^{2\pi} (4 + 3 \cos \theta)^2 \, d\theta = \frac{1}{2} \int_0^{2\pi} (16 + 24 \cos \theta + 9 \cos^2 \theta) \, d\theta
\]

\[
= \frac{1}{2} \int_0^{2\pi} \left( \frac{41}{2} + 24 \cos \theta + \frac{9}{2} \cos 2\theta \right) \, d\theta = \frac{41\pi}{2}
\]

since we integrate through complete periods of cosine. (B)

8. Find the slope of the line tangent to \( x^2 y + 2xy^3 - x^4 = 2 \) at the point \((1, 1)\).

**Solution.** Deriving implicitly, \( 2xy + x^2 \frac{dy}{dx} + 2y^3 + 6xy^2 - 4x^3 = 0 \), so \( \frac{dy}{dx} = -\frac{2xy + 2y^3 - 4x^3}{x^2 + 6xy^2} \).

Plugging in \( x = y = 1 \) gives \( \frac{dy}{dx} = 0 \). (A)

9. Evaluate \( \lim_{x \to 0} \frac{\sin(x^2 \sin(x^2)) + x \sin(\sin(2x))}{\sin(2x) \sin(x \sin(x)) + \sin(\sin(x \sin(x)))} \).

**Solution.** Using that \( \sin(x) = x + O(x^3) \) near \( x = 0 \) can easily give us that the numerator is \((x^4 + O(x^8)) + (2x^2 + O(x^4)) \to 2x^2 \) as \( x \to 0 \) and the denominator is \((2x^3 + O(x^7)) + (x^2 + O(x^4)) \to x^2 \) as \( x \to 0 \). Therefore the ratio is 2 as \( x \to 0 \). (D)

10. Find the volume of the solid that results when the area between the curve \( y = e^{2x} \) and the lines \( y = 0 \), \( x = 1 \), and \( x = 2 \) is rotated around the \( x \)-axis.

**Solution.** \( \pi \int_1^2 \left( (e^{2x})^2 - 0^2 \right) \, dx = \pi \int_1^2 e^{4x} \, dx = \frac{\pi(e^8 - e^4)}{4} \). (C)

11. Evaluate \( \int_{1}^{2021} (x - 1)(x - 2) \cdots (x - 2021) \, dx \).

**Solution.** Let \( u = x - 1011 \). Then we get

\[
\int_{1}^{2021} (x - 1)(x - 2) \cdots (x - 2021) \, dx = \int_{-1010}^{1010} (u + 1010)(u + 1009) \cdots (u - 1010) \, du
\]

\[
= \int_{-1010}^{1010} (u^2 - 1010^2)(u^2 - 1009^2) \cdots (u^2 - 1)^2 \, du.
\]

The integrand is odd and the bounds are symmetric, so the integral has value 0. (D)
12. Determine the convergence or divergence of the infinite series

\[
\frac{1}{\ln(4) \ln(2)} + \frac{1}{\ln(27) \ln(3)} + \frac{1}{\ln(256) \ln(4)} + \frac{1}{\ln(3125) \ln(5)} + \frac{1}{\ln(46656) \ln(6)} + \cdots \nonumber.
\]

**Solution.** The general term of the series is

\[
\frac{1}{\ln(n^n) \ln(n)} = \frac{1}{n \ln(n)}
\]

starting with \(n = 2\). By the integral test,

\[
\int_{2}^{\infty} \frac{1}{x \ln^{2}(x)} \, dx = \lim_{b \to \infty} \left. \frac{-1}{\ln(x)} \right|_{2}^{b} = \lim_{b \to \infty} \left( \frac{-1}{\ln(b)} + \frac{1}{\ln(2)} \right) = \frac{1}{\ln(2)}.
\]

Hence, the series is absolutely convergent. (A)

13. Evaluate \(\int_{1}^{2} \frac{3x^{2} - 4}{x^{3} - 4x + 5} \, dx\).

**Solution.** The numerator is the derivative of the denominator, so \(u = x^{3} - 4x + 5\) gives \(\int_{2}^{5} \ln u \, du = \ln \left( \frac{5}{2} \right)\). (B)

14. Find the length of the polar curve \(r = \sqrt{1 + \cos(2\theta)}\) over the interval \([0, 2\pi]\).

**Solution.** The arc length of a polar curve is given by

\[
\int_{\alpha}^{\beta} \sqrt{(r')^{2} + r^{2}} \, d\theta.
\]

We have \(r = \sqrt{1 + 2 \cos(2\theta)}\) so \(r' = -\frac{1}{2} (1 + \cos(2\theta))^{-1/2} (2 \sin(2\theta)) = \frac{\sin(2\theta)}{\sqrt{1 + \cos(2\theta)}}\). Thus, we compute

\[
\int_{0}^{2\pi} \sqrt{\frac{\sin^{2}(2\theta)}{1 + \cos(2\theta)} + 1 + \cos(2\theta)} \, d\theta = \int_{0}^{2\pi} \sqrt{2} \, d\theta = 2\pi \sqrt{2}.
\]

So the answer is (B).

15. Compute \(\int_{0}^{1} \arctan \sqrt{x} \, dx\).

**Solution.** Let \(u^{2} = x\), so \(2u \, du = dx\). The integral then becomes \(\int_{0}^{1} 2u \arctan u \, du\). Using integration by parts to get the antiderivative results in

\[
u^{2} \arctan u - \int \frac{u^{2}}{1 + u^{2}} \, du = u^{2} \arctan u - \int \left(1 - \frac{1}{1 + u^{2}}\right) \, du
\]

\[
= u^{2} \arctan u - u + \arctan u + C
\]

\[
= (u^{2} + 1) \arctan u - u + C
\]

\[
= (x + 1) \arctan \sqrt{x} - \sqrt{x} + C.
\]

Evaluating from 0 to 1, we get \(2 \arctan 1 - 1 = \pi/2 - 1 = (\pi - 2)/2\). (C)
16. How many continuous functions $f$ with a domain of $[0, 1]$ satisfy this integral equation?
\[ \int_0^1 (f(x))^2 \, dx = \int_0^1 (f(x))^3 \, dx = \int_0^1 (f(x))^4 \, dx \]

Solution. We then have
\[ 0 = \int_0^1 f(x)^2 \, dx - 2 \int_0^1 f(x)^3 \, dx + \int_0^1 f(x)^4 \, dx = \int_0^1 f(x)^2(f(x) - 1)^2 \, dx \]

As $f$ is continuous and the integrand is always nonnegative, it follows that on $[0, 1]$ we have $f(x)^2(f(x) - 1)^2 = 0$, so for each $x \in [0, 1]$ either $f(x) = 0$ or $f(x) = 1$. As $f$ is continuous it must be that $f \equiv 0$ or $f \equiv 1$, giving the answer 2. (C)

17. Find $\frac{d^2y}{dx^2}$ where $x = t^2$ and $y = t^2 + t$.

Solution. We obtain
\[ \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{2t + 1}{2t} = 1 + \frac{1}{2t} \]
and so
\[ \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{1}{2t} \right) = -1/(4t^2) = -\frac{1}{4t^3}. \]

So the answer is (A).

18. A value of $\theta$ is uniformly randomly selected from the range $\left[ \frac{\pi}{6}, \frac{\pi}{4} \right]$. Find the expected value of $\sec^2 \theta$.

Solution. Note that $\frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$. We are looking for $\frac{12}{\pi} \int_{\pi/6}^{\pi/4} \sec^2 \theta \, d\theta = \frac{12}{\pi} \left( \tan \frac{\pi}{4} - \tan \frac{\pi}{6} \right) = \frac{12 - 4\sqrt{3}}{\pi}$. (B)

19. Find $f^{(5)}(0)$, where $f(x) = \arctan(x)$.

Solution. As $f'(x) = 1/(1 + x^2) = 1 - x^2 + x^4 - \cdots$ we seek the fourth derivative, which is just $4!$ times the coefficient of $x^4$ of the Maclaurin series, which is 1. Hence the answer is $4! = 24$. (A)

20. Let $f(x) = x^{\ln(x)}$. Evaluate $f'(2)$.

Solution. Taking the natural logarithm of both sides and differentiating gives
\[ \ln(f(x)) = \ln(x^{\ln(x)}) = (\ln(x))^2 \]
\[ \frac{f'(x)}{f(x)} = \frac{2 \ln(x)}{x} \]
\[ f'(x) = \frac{2f(x) \ln(x)}{x} = \frac{2x^{\ln(x)} \ln(x)}{x}. \]

Hence $f'(2) = \frac{2 \cdot 2^{\ln(2)} \ln(2)}{2} = 2^{\ln(2)} \ln(2)$. (A)
21. A particle moving in the $xy$-plane has acceleration vector $\mathbf{a}(t) = (9t^2 - 4)i + (4t + 1)j$ for all $t \geq 0$, and it has velocity vector $\mathbf{v}(t) = -i - 2j$ at time $t = 0$. What is the speed of the particle at time $t = 2$?

Solution. Integrating the vector $\mathbf{a}(t)$, we get

$$\mathbf{v}(t) = \int \mathbf{a}(t) \, dt = (3t^3 - 4t + C_1)i + (2t^2 + t + C_2)j.$$  

Using the initial value, we find that $C_1 = -1$ and $C_2 = -2$ so that $\mathbf{v}(t) = (3t^3 - 4t - 1)i + (2t^2 + t - 2)j$. At time $t = 2$, we have $\mathbf{v}(2) = 15i + 8j$. The speed of the particle is therefore

$$\sqrt{15^2 + 8^2} = 17.\quad \text{(D)}$$

22. Compute $\int_0^\infty \frac{|x|}{(1 + x)^2} \, dx$ where $|x|$ is the greatest integer less than or equal to $x$.

Solution. We can rewrite this as

$$\sum_{n=0}^\infty \int_n^{n+1} \frac{n}{(1 + x)^2} \, dx = \sum_{n=0}^\infty \frac{n}{1 + x} \bigg|_n^{n+1} = \sum_{n=0}^\infty \frac{n}{n+1} - \frac{n}{n+2} = \sum_{n=0}^\infty \frac{n}{(n+1)(n+2)}$$

which diverges by the limit comparison with the harmonic series. \text{(D)}

23. Let $f(x) = (x^2 + 3)^3$. Evaluate $\frac{dy}{d\sqrt{x}}$ at $x = 1$.

Solution. We have that

$$\frac{dy}{d\sqrt{x}} = \frac{dy}{dx} \frac{dx}{d\sqrt{x}} = \frac{dy}{dx} \frac{1}{d\sqrt{x}/dx} = 6x(x^2 + 3)^2 \frac{1}{1/(2\sqrt{x})} = 12x(x^2 + 3)^2 \sqrt{x}$$

Evaluated at 1, we obtain $12 \cdot 4^2 \cdot 1 = 192.\quad \text{(C)}$

24. Find the $y$-intercept of the tangent line to the curve defined parametrically by $x = e^{3t} + 2$ and $y = \ln(e^{6t} + 4e^{3t} + 4)$ at the point where $t = \ln 2$.

Solution. Note that $y = \ln(e^{6t} + 4e^{3t} + 4) = \ln(e^{3t} + 2)^2 = 2\ln(e^{3t} + 2) = 2\ln(x)$. Hence, we can use the Cartesian equation to find $y'(x) = 2/x$. When $t = \ln 2$, then $x = e^{3\ln 2} + 2 = e^{\ln 2^3} + 2 = 10$ and $y = 2\ln 10$. It follows that the tangent line is $y = 2\ln 10 + (x - 10)/5$, and the $y$-intercept is $y(0) = 2\ln 10 - 2.\quad \text{(D)}$

25. Evaluate $\lim_{x \to 1} \frac{(\sqrt[3]{x} - 1)(\sqrt[3]{x} - 1)(\sqrt[3]{x} - 1)}{(x - 1)^3}$.

Solution. Factoring $x - 1$ as a difference of cubes, fourth powers, and fifth powers yields

$$\frac{(\sqrt[3]{x} - 1)(\sqrt[3]{x} - 1)(\sqrt[3]{x} - 1)}{(x - 1)^3} = \frac{(\sqrt[3]{x} - 1)(\sqrt[3]{x} - 1)(\sqrt[3]{x} - 1)}{(x - 1)(x - 1)(x - 1)}$$

$$= \frac{1}{(x^{2/3} + x^{1/3} + 1)(x^{3/4} + x^{1/4} + 1)(x^{4/5} + x^{3/5} + x^{2/5} + x^{1/5} + 1)}$$

so the limit as $x \to 1$ is $1/(3 \cdot 4 \cdot 5) = 1/60.\quad \text{(B)}$
26. Evaluate \( \lim_{n \to \infty} \frac{1}{n!} \sum_{k=1}^{\infty} \frac{1}{(n+k)!k!} \).

**Solution.** We have

\[
\lim_{n \to \infty} \frac{1}{n!} \sum_{k=1}^{\infty} \frac{1}{(n+k)!k!} = \lim_{n \to \infty} \frac{1}{n!} \sum_{k=1}^{\infty} \frac{k!n!}{(n+k)!k!} = \lim_{n \to \infty} \frac{n!}{n!} \sum_{k=1}^{\infty} \frac{1}{(n+k)!k!} = \lim_{n \to \infty} \frac{n!}{n!} \sum_{k=1}^{\infty} \frac{1}{(n+k)!k!} = \lim_{n \to \infty} \frac{n!}{n!} \sum_{k=1}^{\infty} \frac{1}{(n+k)!k!}
\]

So the answer is (A).

27. Evaluate \( \int_{1}^{e} \frac{x - 1}{x + x^2 \ln(x)} \, dx \).

**Solution.** Divide numerator and denominator by \( x^2 \) to get

\[
\int_{1}^{e} \frac{x - 1}{x + x^2 \ln(x)} \, dx = \int_{1}^{e} \frac{1/x - 1/x^2}{1/x + \ln(x)} \, dx.
\]

Letting \( u = 1/x + \ln(x) \) we get \( du = (-1/x^2 + 1/x) \, dx \) so that

\[
\int_{1}^{e} \frac{1/x - 1/x^2}{1/x + \ln(x)} \, dx = \int_{1}^{1/e} \frac{1}{u} \, du = \ln\left(1 + \frac{1}{e}\right) = \ln\left(\frac{e + 1}{e}\right) = \ln(e + 1) - 1.
\]

So the answer is (B).

28. A particle’s movement in the coordinate plane is parametrized by \( x = \sin^2 \theta \) and \( y = \cos(2\theta) \). Find the total distance (not displacement) the particle travels as \( t \) increases from 0 to \( 2022\pi \).

**Solution.** Note that \( \cos(2\theta) = 1 - 2\sin^2 \theta \), so this is the line segment \( y = 1 - 2x \) as \( x \) ranges from 0 to 1; it has length \( \sqrt{5} \). The particle moves from \((0,1)\) to \((1,-1)\) in the range \( t \in [0, \frac{\pi}{2}] \), and then back in \( t \in [\frac{\pi}{2}, \pi] \), for a total distance of \( 2\sqrt{5} \). This process will repeat 2021 more times, so the total distance the particle moves in \( t \in [0, 2022\pi] \) is \( 4044\sqrt{5} \). (D)

29. Everybody knows l’Hôpital’s rule. But do you know the namesake mathematician’s first name? (Hint: He is French.)

**Solution.** It is Guillaume. (A)

30. Evaluate \( \int 2022x^{2021} \, dx \).

**Solution.** The antiderivative is \( x^{2022} + C \). (B)