For each of the following questions, E) NOTA should be answered when None Of The Answers listed are correct. Throughout, $f^{(n)}(x)$ represents the n^{th} derivative with respect to x, and the 0^{th} derivative is the function itself. Good luck and Have Fun!

(1) If $xy' = y$ and $y(20) = 22$, how many possible values are there for $y(2022)$?

(a) 0 (b) 1 (c) 2

(d) 3 (e) NOTA

SOLUTION: $y' = \frac{y}{x}$ has a singular point at $x = 0$. Since both $x = 20$ and $x = 2022$ are greater than zero, uniqueness holds, so the answer is (b).

(2) If $xy' = y$ and $y(20) = 22$, how many possible values are there for $y(-2022)$?

(a) 0 (b) 1 (c) 2

(d) 3 (e) NOTA

SOLUTION: $y' = \frac{y}{x}$ has a singular point at $x = 0$. Since $x = 20$ and $x = -2022$ are on opposite sides of the singular point, uniqueness does not hold. In fact, any function of the form $y = \begin{cases} \frac{22}{20}x, x > 0 \\ kx\ , x < 0 \end{cases}$ will satisfy both the differential equation and the initial condition. So there are infinite possible values for $y(-2022)$, (e).

(3) If $xy' = y\sin(x)$ and $y(20) = 22$, how many possible values are there for $y(-2022)$?

(a) 0 (b) 1 (c) 2

(d) 3 (e) NOTA

SOLUTION: $y' = \frac{\sin(x)}{x}y$ does not have a singular point at $x = 0$ because $\frac{\sin(x)}{x}$ is analytic. Therefore uniqueness holds, so the answer is (b).

(4) If $yy' = -x$ and $y(20) = 22$, how many possible values are there for $y(2022)$?

(a) 0 (b) 1 (c) 2

(d) 3 (e) NOTA

SOLUTION: This is the familiar differential equation for a circle centered at the origin. The radius of this circle will be $\sqrt{22^2 + 20^2} < 2022$, so there will be no possible values for $y(2022)$. (a).

(5) Find $y(2)$ if $y' = 3x^2 + 2$ and $y(1) = 5$.

(a) 8 (b) 10 (c) 12

(d) 14 (e) NOTA

SOLUTION: $y' = 3x^2 + 2 \to y = x^3 + 2x + C \to 5 = 1 + 2 + C \to C = 2 \to y = x^3 + 2x + 2 \to y(2) = 8 + 4 + 2 = 14$. (d).

(6) Find $y(2\sqrt{2})$ if $y' = x\sqrt{x^2 + 8}$ and $y(1) = 5$.

(a) $\frac{47}{2}$ (b) $\frac{81}{2}$ (c) $\frac{52}{3}$

(d) $\frac{76}{3}$ (e) NOTA

SOLUTION: $y' = x\sqrt{x^2 + 8} \to y = \frac{1}{3}(x^2 + 8)^{\frac{3}{2}} + C \to 5 = \frac{1}{3}(9)^{\frac{3}{2}} + C = 9 + C \to C = -4 \to y(2\sqrt{2}) = \frac{1}{3}(8 + 8)^{\frac{3}{2}} - 4 = \frac{52}{3}$. (c).

(7) Find $y(1)$ if $y' = 22xe^y$ and $y(0) = \ln\left(\frac{1}{22}\right)$.

(a) $-\ln(44)$ (b) $-\ln(22)$ (c) $-\ln(11)$

(d) $-\ln(5.5)$ (e) NOTA

SOLUTION: $y' = 22xe^y \to e^{-y}dy = 22xdx \to -e^{-y} = 11x^2 + C \to -e^{-\ln\left(\frac{1}{22}\right)} = -22 = C \to -e^{-y} = 11x^2 - 22 \to -e^{-y(1)} = 11 - 22 = -11 \to y(1) = -\ln(11)$. (c).

(8) Find $y(2)$ if $y' = 2xy - 2x - y + 1$ and $y(1) = 5$.

(a) $1 + 4e^2$ (b) $1 - 4e^2$ (c) $4e^2 - 1$

(d) $-4e^2 - 1$ (e) NOTA

SOLUTION: $y' = 2xy - 2x - y + 1 = (2x-1)(y-1) \rightarrow \frac{1}{y-1}dy = (2x-1)dx \rightarrow$ $\ln(y-1) = x^2 - x + C \rightarrow y = 1 + De^{x^2-x} \rightarrow 5 = 1 + D \rightarrow D = 4 \rightarrow y(2) = 1 + 4e^2$. (a).

(9) A certain population $P(t)$ of animals is governed by the differential equation $\frac{dP}{dt} = kP(2022 - P)$. If a population with an initial number of 20 animals has 22 animals after one year, find which of the following is closest to $\lim_{t\to\infty} P(t)$.

(a) 0 (b) 20 (c) 22

(d) 2022 (e) NOTA

SOLUTION: The carrying capacity of the population is the limiting value, and is evident from the differential equation as 2022. (d).

(10) A 4-lb roast, initially at 50° F, is placed in a 400° F oven at 5:00 P.M. After 50 minutes it is found that the temperature $T(t)$ of the roast is 150° F. When will the roast be 200° F? Assume the system obeys Newton's Law of Cooling.

(a) $\frac{50\ln\left(\frac{4}{7}\right)}{\ln\left(\frac{5}{7}\right)}$ (b) $\frac{50\ln\left(\frac{5}{7}\right)}{\ln\left(\frac{4}{7}\right)}$ (c) $\frac{25\ln\left(\frac{4}{7}\right)}{2\ln\left(\frac{5}{7}\right)}$

(d) $\frac{25\ln\left(\frac{5}{7}\right)}{2\ln\left(\frac{4}{7}\right)}$ (e) NOTA

SOLUTION: $\frac{dT}{dt} = k(400 - T) \rightarrow \frac{1}{400-T}dT = kdt \rightarrow -\ln(400 - T) = kt + c \rightarrow T = 400 - Ce^{-kt}$. $T(0) = 50 = 400 - C \rightarrow C = 350$. $T(50) = 150 = 400 - 350e^{-50k} \rightarrow k = -\frac{1}{50}\ln\left(\frac{5}{7}\right)$. Therefore $200 = 400 - 350e^{\frac{1}{50}\ln\left(\frac{5}{7}\right)t} \rightarrow \frac{4}{7} = e^{\frac{1}{50}\ln\left(\frac{5}{7}\right)t} \rightarrow t = \frac{50\ln\left(\frac{4}{7}\right)}{\ln\left(\frac{5}{7}\right)}$. (a).

(11) Torricelli's Law states that if a tank with liquid of volume V is draining from a hole in the bottom with area a, then the depth of the water at time t, $y(t)$, is governed by the differential equation $\frac{dV}{dt} = -a\sqrt{2gy}$. Assume $g = 32\ ft/s^2$. A hemispherical bowl has top radius 4 ft and at time $t = 0$ is full of water. At that moment a circular hole with diameter 4 in. is opened in the bottom of the tank. How long will it take (in seconds) for all the water to drain from the tank?

(a) $\frac{168}{5}$ (b) $\frac{224}{5}$ (c) $\frac{336}{5}$

(d) $\frac{672}{5}$ (e) NOTA

SOLUTION: $\frac{dV}{dt} = A(y)\frac{dy}{dt}$, where $A(y)$ is the cross-sectional area at depth y. Using the Pythagorean theorem, $A(y) = \pi(4^2 - (4-y)^2) = \pi(8y - y^2)$. Therefore $\pi(8y - y^2)\frac{dy}{dt} = -8\pi\left(\frac{1}{6}\right)^2\sqrt{y} \rightarrow \left(8y^{\frac{1}{2}} - y^{\frac{3}{2}}\right)\frac{dy}{dt} = -\frac{2}{9} \rightarrow \frac{16}{3}y^{\frac{3}{2}} - \frac{2}{5}y^{\frac{5}{2}} = -\frac{2}{9}t + C$. $y(0) = 4$ so $C = \frac{128}{3} - \frac{64}{5} = \frac{448}{15}$. Therefore, $y(t) = 0$ when $-\frac{2}{9}t + \frac{448}{15} = 0 \rightarrow t = \frac{672}{5}$. (d).

(12) Find $y\left(\frac{\pi}{6}\right)$ if $y' + \tan(x)\,y = 2\sec(x)$ and $y\left(\frac{\pi}{4}\right) = 3\sqrt{2}$.

(a) $1 + 3\sqrt{2}$ (b) $1 + 2\sqrt{3}$ (c) $1 - 3\sqrt{2}$

(d) $1 - 2\sqrt{3}$ (e) NOTA

SOLUTION: $e^{\int \tan(x)dx} = \sec(x) \rightarrow \sec(x)\,y' + \sec(x)\tan(x)\,y = 2\sec^2(x) \rightarrow [\sec(x)\,y]' = 2\sec^2(x) \rightarrow \sec(x)\,y = 2\tan(x) + C \rightarrow y = 2\sin(x) + C\cos(x) \rightarrow 3\sqrt{2} = \sqrt{2} + C\frac{\sqrt{2}}{2} \rightarrow C = 4 \rightarrow y = 2\sin(x) + 4\cos(x) \rightarrow y\left(\frac{\pi}{6}\right) = 1 + 2\sqrt{3}$. (b).

(13) A 120 gal tank initially contains 90 lb of salt dissolved in 90 gal of water. Brine containing 2 lb/gal of salt flows into the tank at the rate of 4 gal/min, and the perfectly stirred mixture flows out of the tank at the rate of 3 gal/min. How much salt (in lbs) does the tank contain when it is full?

(a) $\frac{6465}{8}$ (b) $\frac{6465}{16}$ (c) $\frac{6465}{32}$

(d) $\frac{6465}{64}$ (e) NOTA

SOLUTION: Let $x(t)$ be the amount of salt (in lbs) in the tank at time t. Then $\Delta x = (2)(4)\Delta t - 3\left(\frac{x}{90+4t-3t}\right)\Delta t \rightarrow \frac{dx}{dt} = 8 - \frac{3}{90+t}x \rightarrow \frac{dx}{dt} + \frac{3}{90+t}x = 8$. $e^{\int\frac{3}{90+t}dt} = (90+t)^3 \rightarrow (90+t)^3\frac{dx}{dt} + 3(90+t)^2 x = 8(90+t)^3 \rightarrow [(90+t)^3 x]' = 8(90+t)^3 \rightarrow (90+t)^3 x = 2(90+t)^4 + C$. When $t = 0$, $(90)^3(90) = 2(90)^4 + C \rightarrow C = -(90)^4$. So $(90+t)^3 x = 2(90+t)^4 - (90)^4 \rightarrow x(t) = 2(90+t) - \frac{90^4}{(90+t)^3}$. The tank is full when $90 + 4t - 3t = 90 + t = 120 \rightarrow t = 30 \rightarrow x(30) = 2(120) - \frac{90^4}{(120)^3} = 240 - \frac{1215}{32} = \frac{6465}{32}$. (c).

(14) Which of the following is a possible value of y when $x = 2$ if $(3x^2 + 2xy)y' = x^2 + 6xy + 3y^2$ and $(1,2)$ is a point on the graph of the curve defined by this differential equation?

(a) $-3 + \sqrt{93}$ (b) $3 + \sqrt{93}$ (c) $-3 + \sqrt{97}$

(d) $3+\sqrt{97}$ (e) NOTA

SOLUTION: $y' = \frac{x^2+6xy+3y^2}{3x^2+2xy} = \frac{3\left(\frac{y}{x}\right)^2+6\left(\frac{y}{x}\right)+1}{2\left(\frac{y}{x}\right)+3}$. Let $v = \frac{y}{x} \to xv' + v = y'$. Then $xv' + v = \frac{3v^2+6v+1}{2v+3} \to xv' = \frac{3v^2+6v+1}{2v+3} - v = \frac{v^2+3v+1}{2v+3} \to \frac{2v+3}{v^2+3v+1}v' = \frac{1}{x} \to \ln(v^2+3v+1) = \ln(x) + C \to v^2+3v+1 = Cx = \left(\frac{y}{x}\right)^2 + 3\left(\frac{y}{x}\right) + 1 \to C = 4+6+1 = 11$. Therefore $11(2) = \frac{1}{4}y^2 + \frac{3}{2}y + 1 \to y^2 + 6y - 84 = 0 \to y = \frac{-6\pm\sqrt{372}}{2} = -3+\sqrt{93}$. (a).

(15) If $xy' - 5x^2y + 3y\ln(y) = 0$, $y > 0$, and $y(1) = e^2$, find $y\left(\frac{1}{2}\right)$.

(a) $e^{27/4}$ (b) $e^{29/4}$ (c) $e^{31/4}$

(d) $e^{33/4}$ (e) NOTA

SOLUTION: Let $v = \ln(y) \to yv' = y' \to xyv' - 5x^2y + 3yv = y(xv' - 5x^2 + 3v) = 0 \to xv' - 5x^2 + 3v = 0 \to xv' + 3v = 5x^2 \to x^3v' + 3x^2v = 5x^4 = [x^3v]' \to x^3v = x^5 + C \to \ln(y) = x^2 + \frac{C}{x^3} \to y = e^{x^2+\frac{C}{x^3}} \to e^2 = e^{1+C} \to C = 1 \to y\left(\frac{1}{2}\right) = e^{\frac{1}{4}+8} = e^{33/4}$. (d).

(16) Which of the following is a solution to the following exact differential equation

$$\left(2xy^3\arctan(xy) + \frac{x^2y^4}{1+x^2y^2}\right) + \left(3x^2y^2\arctan(xy) + \frac{x^3y^3}{1+x^2y^2}\right)\frac{dy}{dx} = 0$$

if $(1,1)$ is a point on the graph of the curve defined by this differential equation?

(a) $x^3y^2\arctan(xy) = \frac{\pi}{4}$ (b) $x^2y^2\arctan(xy) = \frac{\pi}{4}$

(c) $x^2y^3\arctan(xy) = \frac{\pi}{4}$ (d) $x^3y^3\arctan(xy) = \frac{\pi}{4}$ (e) NOTA

SOLUTION: This equation is exact, specifically the total derivative of $x^2y^3\arctan(xy) = C \to \arctan(1) = \frac{\pi}{4} = C \to x^2y^3\arctan(xy) = \frac{\pi}{4}$. (c).

(17) Which of the following is a solution to the following differential equation

$$(2x+2y+1) + (3x+3y+1)\frac{dy}{dx} = 0$$

if $y(0) = 1$?

(a) $\ln(x+y) + 2x + 3y = 3$ (b) $\ln(x+y) + 3x + 3y = 3$

(c) $\ln(x+y) + 3x + 2y = 2$ (d) $\ln(x+y) + 2x + 2y = 2$ (e) NOTA

SOLUTION: One way to solve this equation is to note that division by $x+y$ yields $\frac{1+\frac{dy}{dx}}{x+y}+2+3\frac{dy}{dx}=0 \rightarrow \ln(x+y)+2x+3y=C \rightarrow \ln(0+1)+2(0)+3(1)=3=C$. Therefore $\ln(x+y)+2x+3y=3$. (a).

(18) What is the general form of the solution to $y''-y'-12y=0$? Assume y is a function of x.

(a) $y=C_1e^{2x}+C_2e^{-6x}$
(b) $y=C_1e^{-4x}+C_2e^{3x}$
(c) $y=C_1e^{4x}+C_2e^{-3x}$
(d) $y=C_1e^{-2x}+C_2e^{6x}$
(e) NOTA

SOLUTION: Let $y=e^{rx} \rightarrow y'=re^{rx} \rightarrow y''=r^2e^{rx}$. Then $e^{rx}(r^2-r-12)=0=(r-4)(r+3) \rightarrow y=C_1e^{4x}+C_2e^{-3x}$. (c).

(19) What is the general form of the solution to $y''+4y'+13y=0$? Assume y is a function of x.

(a) $y=C_1e^{-2x}\sin(6x)+C_2e^{-2x}\cos(6x)$
(b) $y=C_1e^{2x}\sin(6x)+C_2e^{2x}\cos(6x)$
(c) $y=C_1e^{-5x}+C_2e^{x}$
(d) $y=C_1e^{5x}+C_2e^{-x}$
(e) NOTA

SOLUTION: Let $y=e^{rx} \rightarrow y'=re^{rx} \rightarrow y''=r^2e^{rx}$. Then $e^{rx}(r^2+4r+13)=0 \rightarrow r=\frac{-4\pm\sqrt{16-52}}{2}=-2\pm 6i \rightarrow y=C_1e^{-2x}\sin(6x)+C_2e^{-2x}\cos(6x)$. (a).

(20) What is the general form of the solution to $y'''+3y''+3y'+y=0$? Assume y is a function of x.

(a) $y=C_1e^{-x}+C_2xe^{-x}+C_3x^2e^{-x}$
(b) $y=C_1e^{x}+C_2xe^{x}+C_3x^2e^{x}$
(c) $y=C_1xe^{-x}+C_2x^2e^{-x}+C_3x^3e^{-x}$
(d) $y=C_1xe^{x}+C_2x^2e^{x}+C_3x^3e^{x}$
(e) NOTA

SOLUTION: Let $y=e^{rx} \rightarrow y'=re^{rx} \rightarrow y''=r^2e^{rx}$. Then $e^{rx}(r^3+3r^2+3r+1)=0=(r+1)^3 \rightarrow y=C_1e^{-x}+C_2xe^{-x}+C_3x^2e^{-x}$. (a).

(21) What is the general form of the solution to $4x^2y''-4xy'+3y=0$? Assume y is a function of x.

(a) $y=C_1e^{\frac{x}{2}}+C_2e^{\frac{3x}{2}}$
(b) $y=C_1e^{-\frac{x}{2}}+C_2e^{-\frac{3x}{2}}$

(c) $y = C_1\sqrt{x} + C_2\sqrt{x^3}$ (d) $y = \frac{C_1}{\sqrt{x}} + \frac{C_2}{\sqrt{x^3}}$ (e) NOTA

SOLUTION: Let $y = x^r \to y' = rx^{r-1} \to y'' = r(r-1)x^{r-2}$. Then $x^r(4r(r-1) - 4r + 3) = 0 = 4r^2 - 8r + 3 = (2r-1)(2r-3) \to y = C_1\sqrt{x} + C_2 x\sqrt{x}$. (c).

(22) What is the general form of the solution to $x^2y'' + 5xy' + 4y = 0$? Assume y is a function of x.

(a) $y = C_1x^2 + C_2x^2\ln(x)$ (b) $y = \frac{C_1}{x^2} + \frac{C_2\ln(x)}{x^2}$

(c) $y = C_1e^{2x} + C_2xe^{2x}$ (d) $y = C_1e^{-2x} + C_2xe^{-2x}$ (e) NOTA

SOLUTION: Let $y = x^r \to y' = rx^{r-1} \to y'' = r(r-1)x^{r-2}$. Then $x^r(r(r-1) + 5r + 4) = 0 = r^2 + 4r + 4 = (r+2)^2 \to y = \frac{C_1}{x^2} + \frac{C_2\ln(x)}{x^2}$. (b).

(23) Which of the following is <u>not</u> a possible value of λ which satisfies the boundary value problem below for non-trivial y?

$$x^2y'' + xy' + \lambda y = 0; y(1) = y(e) = 0$$

(a) π^2 (b) $4\pi^2$ (c) $8\pi^2$

(d) $16\pi^2$ (e) NOTA

SOLUTION: Letting $y = x^r, r(r-1)x^r + rx^r + \lambda x^r = 0 \to r^2 + \lambda = 0 \to r = \pm i\sqrt{\lambda} \to y = x^{i\sqrt{\lambda}} = e^{i\sqrt{\lambda}\ln(x)} \to y = C_1\cos\left(\sqrt{\lambda}\ln(x)\right) + C_2\sin\left(\sqrt{\lambda}\ln(x)\right)$. $y(1) = 0 \to C_1 = 0$. $y(e) = 0 \to \sin\left(\sqrt{\lambda}\right) = 0 \to \lambda = n^2\pi^2$. Only (c) is not of this form.

(24) Given that $y = x$ is a solution to $(x^2-1)y'' + xy' - y = 0$, find the solution to this differential equation if $y\left(\sqrt{2}\right) = 0$ and $y'\left(\sqrt{2}\right) = 1$.

(a) $y = \sqrt{x^2-1} - \frac{x}{\sqrt{2}}$ (b) $y = \frac{\sqrt{x^2-1}}{x} - \frac{x}{2}$

(c) $y = \frac{\sqrt{2x^2-2}}{x} - \frac{x}{\sqrt{2}}$ (d) $y = \sqrt{2x^2-2} - x$ (e) NOTA

SOLUTION: Using reduction of order, we set $y = xu \to y' = xu' + u \to y'' = xu'' + 2u'$. Therefore $x^3u'' + 2x^2u' - xu'' - 2u' + x^2u' + xu - xu = 0 \to (x^3 - x)u'' + (3x^2 - 2)u' = 0 \to \frac{u''}{u'} = -\frac{3x^2-2}{x^3-x} = -\frac{3x^2-1}{x^3-x} - \frac{1}{x} + \frac{1}{2}\frac{1}{x+1} + \frac{1}{2}\frac{1}{x-1} \to \ln(u') = -\ln(x^3-x) - \ln(x) + \frac{1}{2}\ln(x+1) + \frac{1}{2}\ln(x-1) = \ln\left(\frac{\sqrt{x^2-1}}{x^4-x^2}\right) \to u' = \frac{\sqrt{x^2-1}}{x^4-x^2} = \frac{1}{x^2\sqrt{x^2-1}} \to u = \frac{\sqrt{x^2-1}}{x} \to y =$

$\sqrt{x^2-1} \rightarrow$ The general solution is $y = C_1x + C_2\sqrt{x^2-1} \rightarrow y(\sqrt{2}) = 0 = C_1\sqrt{2} + C_2$.
$y' = C_1 + \frac{C_2 x}{\sqrt{x^2-1}} \rightarrow y'(\sqrt{2}) = 1 = C_1 + C_2\sqrt{2} \rightarrow C_1 = -1, C_2 = \sqrt{2} \rightarrow y = \sqrt{2x^2-2} - x$. (d).

(25) Let $f(x)$ be a continuous, differentiable function satisfying

$$\left(f(x)\right)^2 = \int_0^x \left(f(t) + f'(t)\right)^2 dt + 2022(1-x)$$

Which of the following is a possible value of $f\left(\frac{\pi}{6}\right)$?

(a) $\frac{\sqrt{674}}{2}$ (b) $-\frac{3\sqrt{2022}}{2}$ (c) $\frac{3\sqrt{674}}{2}$

(d) $-\frac{\sqrt{2022}}{2}$ (e) NOTA

SOLUTION: $\left(f(x)\right)^2 = \int_0^x \left(f(t) + f'(t)\right)^2 dt + 2022(1-x) \rightarrow \left(f(0)\right)^2 = 2022$ and

$\frac{d}{dx}(f(x)^2) = \frac{d}{dx}\left(\int_0^x \left(f(t) + f'(t)\right)^2 dt + 2022(1-x)\right) \rightarrow 2f(x)f'(x) = \left(f(x) + f'(x)\right)^2 - 2022 \rightarrow 2f(x)f'(x) = \left(f(x)\right)^2 + 2f(x)f'(x) + \left(f'(x)\right)^2 - 2022 \rightarrow \left(f(x)\right)^2 + \left(f'(x)\right)^2 = 2022 \rightarrow f(x) = \pm\sqrt{2022}\cos(x)$ or $f(x) = \pm\sqrt{2022}\sin(x)$. $f(0) = \pm\sqrt{2022}$ so the only option is $f(x) = \pm\sqrt{2022}\cos(x)$. Therefore $f\left(\frac{\pi}{6}\right) = \pm\frac{\sqrt{2022*3}}{2} = \pm\frac{3\sqrt{674}}{2}$. (c).

(26) Let y_p be the particular solution to the non-homogenous differential equation

$$y''' + 4y' = 8\sec(2x)$$

Find $y_p(\pi)$.

(a) -2π (b) 2π (c) $-\pi$

(d) π (e) NOTA

SOLUTION: The homogeneous solution is $y_h = C_1 + C_2\cos(2x) + C_3\sin(2x)$. Using variation of parameters, we want to find functions so that

$y_p = y_1u_1 + y_2u_2 + y_3u_3 \rightarrow \begin{bmatrix} 1 & \cos(2x) & \sin(2x) \\ 0 & -2\sin(2x) & 2\cos(2x) \\ 0 & -4\cos(2x) & -4\sin(2x) \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 8\sec(2x) \end{bmatrix} \rightarrow \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} =$

$\frac{1}{8}\begin{bmatrix} \because & \because & 2 \\ \because & \because & -2\cos(2x) \\ \because & \because & -2\sin(2x) \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 8\sec(2x) \end{bmatrix} = \begin{bmatrix} 2\sec(2x) \\ -2 \\ -2\tan(2x) \end{bmatrix} \to \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \ln|\sec(2x)+\tan(2x)| \\ -2x \\ -\ln|\sec(2x)| \end{bmatrix} \to$

$y_p = \ln|\sec(2x)+\tan(2x)| - 2x\cos(2x) - \sin(2x)\ln|\sec(2x)| \to y_p(\pi) = -2\pi.$ (a).

(27) Which of the following is equivalent to $4e^{\begin{bmatrix} 0 & \ln(2) \\ \ln(2) & 0 \end{bmatrix}}$?

(a) $\begin{bmatrix} -5 & -3 \\ -3 & -5 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} -5 & 3 \\ 3 & -5 \end{bmatrix}$ (e) NOTA

SOLUTION: The eigenvalues of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are given by the characteristic equation $l^2 - 1 = 0 \to l = \pm 1$. The eigenvectors associated with these eigenvalues are $\begin{bmatrix} \mp 1 & 1 \\ 1 & \mp 1 \end{bmatrix}\vec{v} = \vec{0} \to \vec{v} = \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix}$. Therefore the solution to the associated differential equation is $\vec{x}(t) = C_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^t + C_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t}$. Therefore the characteristic matrix is $\begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix}$ and $e^{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}t} = \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{bmatrix}$. At $\ln(2)$, this is $\frac{1}{2}\begin{bmatrix} 2+\frac{1}{2} & 2-\frac{1}{2} \\ 2-\frac{1}{2} & 2+\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{bmatrix}$. (b) after multiplication by four.

(28) What is the radius of convergence of the Maclaurin series of the general solution to the differential equation $(x^2 - 6x + 25)y'' + \tan\left(\frac{x}{2022}\right)y' - y = 0$?

(a) 0 (b) 1011π (c) 2022π

(d) 5 (e) NOTA

SOLUTION: This equation will have singular points nearest to the origin when $x^2 - 6x + 25 = 0 \to (x-3)^2 + 16 = 0 \to x = 3 \pm 4i$ and $\frac{x}{2022} = \pm\frac{\pi}{2} \to x = \pm 1011\pi$. The shortest distance to any of these points to the origin is $\sqrt{3^2 + 4^2} = 5$, so that is the radius of convergence. (d).

For the last two questions on this test, you are expected to use the important Differential Equations concept of the Laplace Transform: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt = F(s)$. The below table of Laplace Transforms may be helpful:

$f(t)$	$\mathcal{L}\{f(t)\}\,;s > 0$

$f^{(n)}(t)$	$s^nF(s)-\sum_{k=0}^{n-1} s^{n-1-k}f^{(k)}(0)$
$\int_0^t f(u)du$	$\frac{1}{s}F(s)$
$e^{at}f(t)\ ;a\in\mathbb{R}$	$F(s-a)\ ;\ s>a$
$u(t-a)f(t-a)=\begin{cases}0 & ,t<a\\ f(t-a), & t\geq a\end{cases};a\in\mathbb{R}^+$	$e^{-as}F(s)$
$f(t)*g(t)=\int_0^t f(u)g(t-u)du$	$F(s)G(s)$
$t^nf(t)\ ;n\in\mathbb{Z}$	$(-1)^nF^{(n)}(s)$
$\frac{f(t)}{t}$	$\int_s^\infty F(v)dv$
$t^n\ ;n\in\mathbb{Z}$	$\frac{n!}{s^{n+1}}$

(29) Find $f(6)$ if $f'(t)-u(t-1)f'(t-1)=1$ and $f(0)=0$.

Hint: Consider $\mathcal{L}\{1+[\![t]\!]\}$, where $[\![t]\!]$ is the greatest integer less than or equal to t.

(a) 6 (b) 7 (c) 15

(d) 21 (e) NOTA

SOLUTION: $\mathcal{L}\{1+[\![t]\!]\}=\int_0^\infty e^{-st}(1+[\![t]\!])dt=\sum_{n=0}^\infty\int_n^{n+1}(1+n)e^{-st}dt=\sum_{n=0}^\infty(1+n)\left[-\frac{1}{s}e^{-s(n+1)}+\frac{1}{s}e^{-sn}\right]=\frac{1}{s}(e^s-1)\sum_{n=0}^\infty(n+1)e^{-(n+1)s}=\frac{1}{s}(e^s-1)\frac{e^{-s}}{(1-e^{-s})^2}=\frac{1}{s(1-e^{-s})}$.
So, $\mathcal{L}\{f'(t)-u(t-1)f'(t-1)\}=sF(s)-e^{-s}sF(s)=\frac{1}{s}\rightarrow F(s)=\frac{1}{s}\left(\frac{1}{s(1-e^{-s})}\right)\rightarrow f(t)=\int_0^t(1+[\![u]\!])du\rightarrow f(6)=\int_0^6(1+[\![u]\!])du=1+2+3+4+5+6=21$. (d).

(30) Find $f(3)$ if

$$4\int_0^t u\cdot f'(u)\cdot f'(t-u)du=3\int_0^t u\cdot f(u)\cdot f''(t-u)du$$

and $f(0)=f'(0)=0$ and $f(1)=1$.

(a) 3 (b) 9 (c) 27

(d) 81 (e) NOTA

SOLUTION:
$4\int_0^t u \cdot f'(u) \cdot f'(t-u)du = 3\int_0^t u \cdot f(u) \cdot f''(t-u)du \to 4[tf'(t)] * [f'(t)] = 3[tf(t)] * [f''(t)] \to \mathcal{L}\{4[tf'(t)] * [f'(t)]\} = \mathcal{L}\{3[tf(t)] * [f''(t)]\} \to 4\mathcal{L}\{[tf'(t)]\}\mathcal{L}\{[f'(t)]\} = 3\mathcal{L}\{[tf(t)]\}\mathcal{L}\{[f''(t)]\} \to 4\left(-\frac{d}{ds}[sF(s)]\right)\left(sF(s)\right) = 3\left(-\frac{d}{ds}F(s)\right)\left(s^2F(s)\right) \to 4(-F - sF')(sF) = 3(-F')(s^2F) \to -4sF^2 - 4s^2FF' = -3s^2FF' \to -4sF^2 = s^2FF' \to -4F = sF' \to -\frac{4}{s} = \frac{F'}{F} \to -4\ln(s) = \ln(F) + C \to F = C\frac{1}{s^4} \to f(t) = Ct^3 \to f(1) = 1 \to C = 1 \to f(t) = t^3 \to f(3) = 27.$ (c).

ANSWERS

1. **B**
2. **E**

3. **B**
4. **A**
5. **D**
6. **C**
7. **C**
8. **A**
9. **D**
10. **A**
11. **D**
12. **B**
13. **C**
14. **A**
15. **D**
16. **C**
17. **A**
18. **C**
19. **A**
20. **A**
21. **C**
22. **B**
23. **C**
24. **D**
25. **C**
26. **A**
27. **B**
28. **D**
29. **D**
30. **C**