Mu Gemini Answers and Solutions

ANSWERS :

1. **E (0)**
2. **A**
3. **B**
4. **B**
5. **C**
6. **D**
7. **D**
8. **A**
9. **D**
10. **C**
11. **D**
12. **A**
13. **B**
14. **B**
15. **D**
16. **E**
17. **C**
18. **D**
19. **D**
20. **C**
21. **A**
22. **C**
23. **B**
24. **C**
25. **B**
26. **A**
27. **A**
28. **B**
29. **E (e^2)**
30. **D**

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SOLUTIONS :

1. The expression within the brackets is a constant (e^{-27} specifically), and the derivative of a constant is 0. **E.**

2. We will use Pappus's Theorem in reverse. For rotation about the x axis: $V = 2\pi dA \to 4 = 2\pi da \to d = \frac{2}{\pi a}$, and therefore q is $\frac{2}{\pi a}$. For rotation about the y axis: $V = 2\pi dA \to 8 = 2\pi da \to d = \frac{4}{\pi a}$. Although the distance from the centroid to the x axis is $\frac{4}{\pi a}$, p is $-\frac{4}{\pi a}$ because the region lies fully in the second quadrant. $\frac{p}{q} = \frac{-\frac{4}{\pi a}}{\frac{2}{\pi a}} = -2$. **A**

3. Let O be the center of both circles, A be the point Andy occupies, and B be the point Buffy occupies. Let θ be the smaller angle formed by rays OA and OB. By the Law of Cosines, $AB^2 = 5^2 + 8^2 - 2 \cdot 5 \cdot 8 \cdot \cos(\theta) = 119 - 80\cos(\theta)$. Differentiating with respect to time gives $2AB\frac{d(AB)}{dt} = 80\sin(\theta)\frac{d\theta}{dt}$. Since Andy and Buffy are both moving around their circles at a constant 2π radians per minute, it follows that $\frac{d\theta}{dt} = 4\pi$ radians per minute. Since $AB = 7$ and it's not hard to find that when $AB = 7, \theta = 60°$. This means $14\frac{d(AB)}{dt} = 40\sqrt{3} \cdot 4\pi$, so $\frac{d(AB)}{dt} = \frac{80\pi\sqrt{3}}{7}$ units per minute. **B.**

4. f is continuous at $x = 0$ because $1 = f(0) = \lim_{x \to 0}\frac{\sin(x)}{x}$. g is not continuous at 0 because the limit from the left is -1, but $g(0) = 1$. h is discontinuous for the same reason; the limit from the left is 1 but $h(0) = 0$. Only f is continuous, so the answer is 1. **B.**

5. First of all, $h(x) = x$ because $f(x)$ and $g(x)$ are inverses. Therefore, $j(x) = \frac{x}{g(x)}$. We are looking for $j'(x)$ at $x = 10 : j'(x) = \frac{[g(x)](1) - x[g'(x)]}{[g(x)]^2}$. Now we will find the numbers to substitute in. $g(10): 10 = y^3 + y \to y = 2$. $g'(10) = \frac{1}{f'(g(10))} = \frac{1}{f'(2)} = \frac{1}{3(2)^2+1} = \frac{1}{13}$. $j'(10) = \frac{(2)(1)-(10)(\frac{1}{13})}{(2)^2} = \frac{4}{13}$. **C.**

6. $\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}(r_1)^2 - (r_2)^2 d\vartheta = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\pi d\vartheta = \frac{\pi}{2}\vartheta\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = \frac{\pi^2}{2}$. **D.**

7. These curves intersect when $2x^3 - 4x^2 - x + 5 = x^3 + 2x^2 - 6x - 7 \to x^3 - 6x^2 + 5x + 12 = 0 \to x = -1, 3, 4$. Since there are two enclosed areas and which functions are on the "top" and "bottom" flips between them, we must break up our integral. $\int_{-1}^{3} x^3 - 6x^2 + 5x + 12\, dx - \int_{3}^{4} x^3 - 6x^2 + 5x + 12\, dx = \left[\frac{99}{4} - \left(-\frac{29}{4}\right)\right] - \left[24 - \frac{99}{4}\right] = 32.75$. **D**

8. The rubber bands are straight line segments. Therefore, to make a smooth curve such as a circle, we will need infinitely many ($n = \infty$) rubber bands to cut each other off into infinitesimally shorts segments to form the circumference of the circle. Since these infinitesimally shorts segments are really just points, the rubber bands are tangent to the circle at these points (which are also their midpoints). They are line segments, not lines, because they have finite length. The derivative of the circle at the point of tangency is represented by the "slope" of the rubber band, not by the rubber band itself. **A.**

9. We have from Question #8 that $n = \infty$. Let's call the angle formed by $Nail\ k$, the center of the board, and $Nail\ [(k+f) mod\ n]$ (with the center of the board being the vertex of the angle) angle θ. If we can find the ratio $\frac{\theta}{2\pi}$, we can set it equal to the ratio of the number of nails between and including one of $Nail\ k$ and $Nail\ [(k+f) mod\ n]$ (which is f) to the total number of nails (which is n). In other words, $\frac{\theta}{2\pi} = \frac{f}{n} \rightarrow f = \frac{\theta n}{2\pi}$. Since any given rubber bands we want to place is tangent to the inner circle, we can draw a right triangle with hypotenuse length R, a leg of length r, and an angle between them of $\frac{\theta}{2}$. Therefore, $\frac{\theta}{2} = \cos^{-1}(\frac{r}{R}) \rightarrow \theta = 2\cos^{-1}(\frac{r}{R})$. By substitution, $f = \frac{2\cos^{-1}(\frac{r}{R})n}{2\pi} = \frac{n\cos^{-1}(\frac{r}{R})}{\pi}$. The negation of this would also work for f because the labeling of the two nails in any appropriate pair is arbitrary. **D.**

10. We will examine a line that is tangent to $f(x)$ at the $x = c$ for some constant c in the domain ($0 \le c \le s$). The tangent line passes through the point $(c, f(c))$ and has slope $f'(c)$. Thus, the line can be written as $y - f(c) = f'(c)(x - c)$. Our situation also tells us that the non-zero coordinates of the x and y-intercepts of our line sum to s. This is because the nails joined by the rubber band that is a segment of the tangent line are the points of interception, and their transformation is equivalent to the aforementioned property. We can now combine our lines of reasoning. The x-coordinate of the x-intercept of our line is $-\frac{f(c)}{f'(c)} + c$, and the y-coordinate of the y-intercept of our line is $-cf'(c) + f(c)$. Therefore, their sum is $-\frac{f(c)}{f'(c)} + c - cf'(c) + f(c)$, which can also be written as $f(c)\left(1 - \frac{1}{f'(c)}\right) + c\left(1 - f'(c)\right)$. Remember that this value is equal to s, so for any c in the domain, including $\frac{s}{3}$, the answer is s. **C.**

11. $\frac{dr}{d\vartheta} = -4\sin(2\vartheta)$. $\frac{dy}{dx} = \frac{\frac{dr}{d\vartheta}\sin(\vartheta) + r cos(\theta)}{\frac{dr}{d\vartheta}\cos(\vartheta) - r sin(\theta)} = \frac{[-4\sin(2\vartheta)]\sin(\vartheta) + r cos(\theta)}{[-4\sin(2\vartheta)]\cos(\vartheta) - r sin(\theta)} =$
$\frac{\left[-4\sin\left(\frac{\pi}{3}\right)\right]\sin\left(\frac{\pi}{6}\right) + [2\cos(\frac{\pi}{3})] cos(\frac{\pi}{6})}{[-4\sin\left(\frac{\pi}{3}\right)]\cos\left(\frac{\pi}{6}\right) - [2\cos(\frac{\pi}{3})] sin(\frac{\pi}{6})} = \frac{\left[-4(\frac{\sqrt{3}}{2})\right]\frac{1}{2} + [2(\frac{1}{2})](\frac{\sqrt{3}}{2})}{[-4(\frac{\sqrt{3}}{2})](\frac{\sqrt{3}}{2}) - [2(\frac{1}{2})]\frac{1}{2}} = \frac{-\sqrt{3} + \frac{\sqrt{3}}{2}}{-3 - \frac{1}{2}} = \frac{-2\sqrt{3} + \sqrt{3}}{-6-1} = \frac{-\sqrt{3}}{-7} = \frac{\sqrt{3}}{7}$. **D.**

12. The volumes of each step can each be derived by drawing pictures and using optimization. However, they are fairly common and can be memorized: $(1)\left(\frac{27}{8}\right)(2)\left(\frac{27}{4}\right)(3)\left(\sqrt{3}\right) = 2^{-4}3^{\frac{15}{2}} \rightarrow$ $ad + bc = (2)\left(\frac{15}{2}\right) + (3)(-4) = 15 - 12 = 3$. **A.**

13. We'll call w the amount of time in *minutes* that Kira spends writing and s her score. $s = w * 2^{\frac{60-w}{5}} \to s' = (w)\left[\left(-\frac{1}{5}\right)\ln(2)\, 2^{\frac{60-w}{5}}\right] + 2^{\frac{60-w}{5}}(1) = 2^{\frac{60-w}{5}}[1 - w\left(\frac{\ln(2)}{5}\right)]$. Setting this equal to 0, we find that a minimum occurs at $w = \frac{5}{\ln(2)}$, where the derivative changes from positive to negative. **B.**

14. Rewrite $\frac{x\coth(x) - 1}{x^2}$ as $\frac{xe^{2x}+x-e^{2x}+1}{x^2e^{2x}-x^2}$ and let $f(x) = xe^{2x} + x - e^{2x} + 1$ and $g(x) = x^2e^{2x} - x^2$. Then $f'(x) = 2xe^{2x} - e^{2x} + 1, f''(x) = 4xe^{2x}$, and $f'''(x) = 8xe^{2x} + 4e^{2x}$. More computation gives $g'(x) = 2x^2e^{2x} + 2xe^{2x} - 2x, g''(x) = 4x^2e^{2x} + 8xe^{2x} + 2e^{2x} - 2$, and $g'''(x) = 8x^2e^{2x} + 24xe^{2x} + 12e^{2x}$. Since $0 = f(0) = f'(0) = f''(0) = g(0) = g'(0) = g''(0)$, by L'Hopital we have $\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'''(x)}{g'''(x)} = \frac{4}{12} = \frac{1}{3}$. **B.**

15. The other statements would be correct it they said: A) $f(x)$ represents the larger radius of the washer. B) $g(x)$ represents the smaller radius of the washer. C) π is part of an expression to represent area(/area of the base) of each washer. **D.**

16. The other statements would be correct it they said: A) x represents the radius of the each cylindrical shell. B) $f(x)$ represents the height of each cylindrical shell . C) 2π is part of an expression to represent the circumferece of each cylindrical shell. D) a and b represent the radii of the smallest and largest radii of the cylindrical shells, respectively. **E.**

17. I is equal to zero because arctan is bounded above by $\frac{\pi}{2}$ while $\ln(x)$ is unbounded. II is equal to 1 because $\lim_{x\to\infty}(1 + e^{-x}) = 1$. III is equal to $\ln(2)$ after noticing that it is equivalent to $\lim_{x\to 0^+}\frac{2^x-1}{x}$ and applying L'Hopital. The expression in IV is unbounded by L'Hopital. Thus the answer is II and III only. **C.**

18. These terms can each be memorized, or they can be derived by drawing a right triangle with legs t opposite of and 1 adjacent to an angle of measure $\frac{x}{2}$. Then, you can use double angle formulas and the picture to find that $cos(x) = \frac{1-t^2}{1+t^2}$ and $sin(x) = \frac{2t}{1+t^2}$. For dx, we can just differentiate $x = 2arctan(t)$ with respect to t: $dx = \frac{2}{1+t^2}dt \to \frac{dx}{dt} = \frac{2}{1+t^2}$. Final answer: $\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} + \frac{2}{1+t^2} = \frac{-t^2+2t+3}{1+t^2}$. **D.**

19. Differentiating both sides of the equation gives: $f'(d) = \frac{1}{\ln(2)} * \frac{1}{\frac{10}{d}} * \frac{-10}{d^2} * \frac{dd}{dt}$. After 3 *seconds*, Alice's finger is at position $d = 3\ seconds * 2\ \frac{inches}{second} = 6\ inches$. So, we want $f'(2)$, and we have $d = 6$ and $\frac{dd}{dt} = -2$ because Alice's finger is moving closer to the bridge. Plugging in our values yields: $f'(2) = \frac{1}{\ln(2)} * \frac{1}{\frac{10}{2}} * \frac{-10}{2^2} * (-2) = \frac{1}{\ln(2)}$. **D.**

20. If we multiply the top and bottom of our integrand by x^{-a}, we get: $\int \frac{1}{x+x^a} * \frac{x^{-a}}{x^{-a}} dx = \int \frac{x^{-a}}{x^{1-a}+1} dx \rightarrow substitute\ u = x^{1-a}+1, du = (1-a)x^{-a}dx \rightarrow \frac{1}{1-a}\int \frac{1}{u} du = \frac{1}{1-a}\ln(|u|) + C = \frac{1}{1-a}\ln(|x^{1-a}+1|) + C$. We can omit the absolute value bars because the quantity within them is positive for all x in our positive domain. $\frac{1}{1-a}\ln(x^{1-a}+1) + C$. $m - 2n - 3p = (1-a) - 2(1-a) - 3(1) = a - 4$. **C.**

21. The volume is composed of three parts: 1 sphere of radius 3 ($V = 36\pi$), n "half-cylinders" of radius 3 and length 10 ($V = 45\pi n$), and 1 prism of base area $\frac{1}{4} * n * 10^2 * \cot\left(\frac{\pi}{n}\right)$ and height $2 * 3$ ($V = 150n\cot(\frac{\pi}{n})$). $f(n) = 36\pi + 45\pi n + 150n\cot\left(\frac{\pi}{n}\right) \rightarrow f'(n) = 45\pi + 150\left(\frac{\pi\csc^2\left(\frac{\pi}{n}\right)}{n} + \cot\left(\frac{\pi}{n}\right)\right) \rightarrow f'(6) = 45\pi + 150\left(\frac{\pi\csc^2\left(\frac{\pi}{6}\right)}{6} + \cot\left(\frac{\pi}{6}\right)\right) = 45\pi + 100\pi + 150\sqrt{3}$. $145 + 150 + 3 = 298$. **A.**

22.

x	1	$\frac{3}{2}$	2
$f(x)$	4	$\frac{15}{2}$	**15**
$\frac{dy}{dx}$	7	15	
dy, which is $\frac{dy}{dx} * \frac{2-1}{2}$	$\frac{7}{2}$	$\frac{15}{2}$	

$\int_0^{15} \frac{3t+4}{t+1} dt = \int_0^{15} 3 + \frac{1}{t+1} dt = 3t + \ln(t+1)\,|_0^{15} = 45 + 4\ln(2)$. **C.**

23. What each test can prove:

	Convergence	Divergence
Nth Term Test	0	1
Integral Test	1	1
P Series Test	1	1
Direct Comparison Test	1	1
Limit Comparison Test	1	1
Alternating Series Theorem	1	0
Ratio test	1	1
	$n = 6$	$m = 6$

$6th$ degree Taylor polynomial for e^x: $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} \rightarrow e^{6-3} = e^3 \approx 1 + 3 + \frac{(3)^2}{2!} + \frac{(3)^3}{3!} + \frac{(3)^4}{4!} + \frac{(3)^5}{5!} + \frac{(3)^6}{6!} = \frac{1553}{80}$. **B.**

24. First we will combine the restriction with the point (0,1). The maximum value we could achieve at $x = 2$ would be when $f'(t) = 4t$ at all times. $1 + \int_0^2 4tdt = 9$. The minimum value we could achieve at $x = 2$ would be when $f'(t) = -4t$ at all times. $1 + \int_0^2 -4tdt = -7$. Next we will combine the restriction with the point (3,14). The maximum value we could achieve at $x = 2$ would be when $f'(t) = -4t$ at all times. $14 + \int_3^2 -4tdt = 24$. The minimum value we could achieve at $x = 2$ would be when $f'(t) = 4t$ at all times. $14 + \int_3^2 4tdt = 4$. Since we must be able to "get to" our point from both sides, we need the intersection of $[-7,9]$ and $[4,24]$, which is $[4,9]$.
$\int_4^9 \frac{1}{2x^{\frac{3}{2}} - 4x + 4x^{\frac{1}{2}}} dx = \int_4^9 \frac{1}{(2\sqrt{x})(2 - 2\sqrt{x} + x)} dx \to substitute\ u = \sqrt{x}, du = \frac{1}{2\sqrt{x}} dx \to \int_2^3 \frac{1}{u^2 - 2u + 2} du = \int_2^3 \frac{1}{(u-1)^2 + 1} du \to substitute\ w = u - 1, dw = du \to \int_1^2 \frac{1}{w^2 + 1} dw = \arctan(w)|_1^2 = \arctan(2) - \frac{\pi}{4}$. **C.**

25. While this volume could be found using integration, we can also use ratios. If the cross sections were semicircular (each having an area of $\frac{\pi x^2}{2}$, where x equals half the length of the relevant chord), the volume would be $\frac{2}{3}\pi(4^3) = \frac{128\pi}{3}$. However, each cross section has an area of $3(\frac{x^2\sqrt{3}}{4})$.
Therefore, the volume is: $\frac{128\pi}{3} * \frac{3\left(\frac{\sqrt{3}}{4}\right)}{\frac{\pi}{2}} = 64\sqrt{3}$. $M = 256 + 33 - 1 = 288$.
$0 + \int_0^x t^3 + 2t - 4dt - \int_2^x 1dt = 62 \to \frac{x^4}{4} + x^2 - 4x - x + 2 = 62 \to x = 4$. **B.**

26. $t = 5x + \frac{845}{3+2x} \to dt = 5 - \frac{845}{(3+2x)^2} \to relative\ minimum\ at\ x = 5$. $V = \frac{\pi(5)^2(3(10)-5)}{3} = \frac{625\pi}{3}$. **A.**

27. Using Pappus's Theorem: $V_{jello} = 2\pi r A = 2\pi\left(2 + \left(\frac{1}{3}\right)2\right)\left[\left(\frac{1}{2}\right)(1)(2)\right] = \frac{16\pi}{3}$.
$V_{cone} = \frac{1}{3}h\left(\pi\left(\sqrt{2}h\right)^2\right) = \frac{2\pi h^3}{3}$. When the volume is $\frac{16\pi}{3}$, $h = 2$. $dV_{cone} = Adh \to 16\pi = \pi\left(2\sqrt{2}\right)^2 dh$. $dh = 2$. **A.**

28. $\int\ sec(x)dx = \int\ sec(x) * \frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)} dx = \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)} dx \to substitute\ u = \sec(x) + \tan(x), du = (\sec(x)\tan(x) + sec^2(x))dx \to \int \frac{1}{u} du = ln|u| + C = ln|\sec(x) + \tan(x)| + C$.
For the second part we will use integration by parts (a variant on product rule) with $u = \sec(x)$ and $dv = \sec^2(x)dx$. Therefore, $du = \sec(x)\tan(x)$ and $v = \tan(x)$.
$\int sec^3(x)dx = \sec(x)\tan(x) - \int \sec(x)\tan^2(x)dx$. We will rewrite the second integral using a trig identity. $\int sec^3(x)dx = \sec(x)\tan(x) - \int \sec(x)(\sec^2(x) - 1)dx = \sec(x)\tan(x) - \int \sec^3(x) - \sec(x)dx$. We can add $\int sec^3(x)dx$ to both sides. $2\int sec^3(x)dx = \sec(x)\tan(x) + \int \sec(x)dx = \sec(x)\tan(x) + ln|\sec(x) + \tan(x)| + C$ (*see part A*). Therefore, the answer to this part is $\frac{\sec(x)\tan(x)}{2} + \frac{ln|\sec(x)+\tan(x)|}{2} + C$.
Final answer: $\frac{\sec(x)\tan(x)+3ln|\sec(x)+\tan(x)|}{2} + C$. **B.**

29. This is a Riemann sum: $\lim_{n\to\infty}\sum_{i=0}^{n}(\sqrt[n]{\frac{e^i}{n^n}} + \frac{1}{n}) = \lim_{n\to\infty}\sum_{i=0}^{n}(\frac{e^{\frac{i}{n}}}{n} + \frac{1}{n}) = \int_0^1 e^x + 1dx = e^x + x|_0^1 = e$.
$\lim_{h\to 0}\frac{L^{2+h} - L^2}{h} = \frac{d}{dx}e^x\ at\ x = 2$, which is e^2. **E.**

30. The answers are "MacLaurin" and "absolutely," so $a = 9$ and $b = 10$. $2ab = 2^2 * 3^2 * 5 \rightarrow (3)(3)(2) = 18$ factors. **D.**