1 Answers

1. A
2. A
3. C
4. B
5. D
6. A
7. C
8. B
9. B
10. E
11. B
12. A
13. B
14. B
15. B
16. B
17. D
18. A
19. D
20. A
21. C
22. A
23. C
24. D
25. E
26. C
27. D
28. C
29. A
30. D

2 Solutions

1. Compute $\int_0^{\pi} \sin x \cos x \; dx$.

   Solution. Let $u = \sin x, du = \cos x \; dx$ and the integral in x becomes an integral in u from 0 to 0 of $u \; du$, which is 0. (A)

2. Compute $\int_0^2 \frac{x}{2+x} \; dx$.

   Solution. Note that we can divide the fraction to yield

$$\int_0^2 \frac{x}{2+x} \; dx = \int_0^2 \left(1 - \frac{2}{x+2}\right) \; dx = x - 2\ln(x+2)\Big|_0^2 = 2 - 2\ln 4 + 2\ln 2 = 2 - \ln 4.$$

   So the answer is (A).

3. Compute $\int_0^{4\pi} |\sin 2x| \; dx$.

   Solution. Note that

$$\int_0^{\pi/2} \sin 2x \; dx = -\frac{1}{2}\cos 2x\Big|_0^{\pi/2} = \frac{1}{2} + \frac{1}{2} = 1.$$

   The interval from 0 to 4π can be broken down into 8 intervals of length $\pi/2$ where all values, and therefore all areas, are positive. Because $\sin 2x$ is periodic, the value is $1 \cdot 8 = 8$. (C)

4. Using a trapezoidal sum on 6 equal subintervals, estimate the value of $\int_2^{20} (4x+1) \; dx$.

   Solution. A trapezoidal sum is exact for linear functions, so we can evaluate the integral directly to get $2 \cdot 20^2 + 20 - 2 \cdot 2^2 - 2 = 810$. (B)

5. What is the total area bounded by the graph of $f(x) = x^3$ and its inverse $f^{-1}(x)$?

   Solution. Two separate regions (between -1 and 0, and 0 and 1) are bounded, and by symmetry with respect to $y = x$, the areas of each region are equal. The total bounded area is thus

$$2\int_0^1 (x^{1/3} - x^3) \; dx = 2\left(\frac{3}{4}x^{4/3} - \frac{1}{4}x^4\right)\Big|_0^1 = 2\left(\frac{3}{4} - \frac{1}{4}\right) = 1.$$

   So the answer is (D).

6. Compute $\lim_{n\to\infty} \sum_{i=0}^{n} \frac{i}{i^2+n^2}$.

   Solution. Dividing each term by n^2, we can rewrite the summand as

$$\frac{i/n^2}{i^2/n^2 + n^2/n^2} = \frac{i/n \cdot 1/n}{(i/n)^2+1} = \frac{1}{n} \cdot \frac{i/n}{(i/n)^2+1}.$$

By the Riemann definition of an integral, this sum is equal to

$$\int_0^1 \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)\Big|_0^1 = \frac{\ln 2}{2}.$$

So the answer is (A).

7. What is the length of the polar curve $r = \theta^2$ from $\theta = 0$ to $\theta = 2$?

   Solution. By the polar arc length definition, we integrate the square root of $r^2 + (r')^2$. Thus, we have that the length of this curve is

$$\int_0^2 \sqrt{\theta^4 + 4\theta^2}\,d\theta = \int_0^2 \theta\sqrt{\theta^2+4}\,d\theta = \frac{1}{3}(\theta^2+4)^{3/2}\Big|_0^2 = \frac{8^{3/2} - 4^{3/2}}{3} = \frac{16\sqrt{2}-8}{3}.$$

   So the answer is (C).

8. If $g(x)$ is an even function and $\int_{\mathbb{R}} g(x)\,dx = 4$ then compute the value of $\int_0^\infty g(x)\,dx$.

   Solution. Because the integral in question is over all positive real numbers and our function is even, this is simply equal to $4/2 = 2$. (B)

9. $e^x(\cos x - \sin x)$ is the derivative of which of the following?

   Solution. The given function is the derivative of $e^x \cos x$ through the product rule. (B)

10. Compute $\displaystyle\int_0^\infty \sum_{n=0}^{\infty} (-x^2)^n\,dx$.

    Solution. The inner summand evaluates to $1/(1+x^2)$ by the sum of an infinite geometric series, provided $0 < x < 1$. But because x goes from 0 to ∞, this integral diverges. (E)

11. The function $f(x) = kx(1-x)^3$ defines a probability density function on $[0,1]$ for some real k. Compute k.

    Solution. To solve for k, we note that $\int_0^1 kx(1-x)^3\,dx = 1$ in order for this to be a probability distribution. Hence, using integration by parts with $u = x$ and $dv = (1-x)^3$, we obtain

$$\begin{aligned} \int_0^1 kx(1-x)^3\,dx &= 1 \\ -\frac{kx}{4}(1-x)^4 - \frac{k}{20}(1-x)^5\Big|_0^1 &= 1 \\ \frac{k}{20} &= 1, \end{aligned}$$

    so $k = 20$. (B)

12. Let R be the region bounded by the parametric equations $x(t) = 2t$ and $y(t) = t/(t^2+1)$ and the x axis over the interval $t \in [0,1]$. What is the area of R?

Solution. The area is given by $\int y\ dx$, which becomes $\int_0^1 2t/(t^2+1)\ dt$ by plugging in $y(t)$ and computing $dx = 2\ dt$. This integral is easily solved through u-substitution and is equal to $\ln 2$. (A)

13. Compute $\displaystyle\int_0^\infty \frac{x^2}{(x^2+1)^2}\ dx$.

    Solution. Let $x = \tan\theta$ so that $dx = \sec^2\theta\ d\theta$. Then the integral becomes

$$\int_0^{\pi/2} \sin^2\theta\ d\theta = \int_0^{\pi/2} \frac{1}{2}(1-\cos 2\theta)\ d\theta = \frac{1}{4}(2\theta - \sin 2\theta)\Big|_0^{\pi/2} = \frac{\pi}{4}.$$

So the answer is (B).

14. Compute $\displaystyle\int_1^\infty \frac{\ln^2 x}{x^2}\ dx$.

    Solution. Let $u = \ln x$ so that $e^u\ du = dx$. The integral then becomes, using integration by parts,

$$\int_0^\infty u^2 e^{-u}\ du = \lim_{b\to\infty} -(u^2+2u+2)e^{-u}\Big|_0^b = 2.$$

So the answer is (B).

15. Compute $\displaystyle\int_0^{2\pi} \sin(\sin x - x)\ dx$.

    Solution. Use sine addition on the integrand to rewrite it as $\sin(\sin x)\cos x - \cos(\sin x)\sin x$. Now we integrate each term. The integral of $\sin(\sin x)\cos x$ over $[0, 2\pi]$ is equal to 0, which can be seen through the u-substitution $u = \sin x$:

$$\int_0^{2\pi} \sin(\sin x)\cos x\ dx = -\cos(\sin x)\Big|_0^{2\pi} = -\cos(0) + \cos(2\pi) = 0.$$

As for the second term, we use the u-substitution $x = 2\pi - u$ and the identity $\sin(2\pi - u) = -\sin u$:

$$\begin{aligned}\int_0^{2\pi} -\cos(\sin x)\sin x\ dx &= \int_{2\pi}^0 \cos(\sin(2\pi - u))\sin(2\pi - u)\ du \\ &= -\int_{2\pi}^0 \cos(-\sin u)\sin u\ du = \int_0^{2\pi} \cos(\sin u)\sin u\ du.\end{aligned}$$

Thus, this transformation takes $\cos(\sin x)\sin(x)$ to $-\cos(\sin x)\sin x$, so this integral must also be 0. Therefore, the entire integral is 0. (B)

16. Compute $\displaystyle\int_0^1 x\ln(1-x)\ dx$.

    Solution. We can write (on the interval $(0,1)$)

$$\ln(1-x) = -\sum_{i=1}^\infty \frac{x^i}{i}, \qquad \text{so that} \qquad x\ln(1-x) = -\sum_{i=1}^\infty \frac{x^{i+1}}{i}.$$

Therefore, we can integrate the series:

$$\int_0^1 x\ln(1-x)\,dx = -\int_0^1 \sum_{i=1}^{\infty} \frac{x^{i+1}}{i} = -\sum_{i=1}^{\infty} \frac{x^{i+2}}{i(i+2)}\Bigg|_0^1 = -\sum_{i=1}^{\infty} \frac{1}{i(i+2)} = -\sum_{i=1}^{\infty} \frac{1}{2}\left(\frac{1}{i} - \frac{1}{i+2}\right).$$

This series telescopes, leaving $-(1/2)(1+1/2) = -3/4$. (B)

17. Compute $\displaystyle\int_0^4 (x-7)(x-2)^5\ dx$.

Solution. Let $u = x - 2$. The integral becomes

$$\int_{-2}^{2} (u-5)u^5\ du = \int_{-2}^{2} (u^6 - 5u^5)\ du.$$

Because the integral of the odd function $-5u^5$ is 0 over $[-2, 2]$, the integral is simply $2 \cdot 2^7/7 = 2 \cdot 128/7 = 256/7$. (D)

18. Compute $\displaystyle\int_0^1 \frac{1}{1+x+x^2}\ dx$.

Solution. We can complete the square on the integrand to rewrite it as

$$\frac{1}{(x+1/2)^2 + 3/4} = \frac{4/3}{((2x+1)/\sqrt{3})^2 + 1}.$$

Let $u = (2x+1)/\sqrt{3}$. Then $du = 2/\sqrt{3}\ dx$ to get that this integral is

$$\frac{4\sqrt{3}}{6}\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{1}{1+u^2}\ du = \frac{2\sqrt{3}}{3}\arctan u\Bigg|_{1/\sqrt{3}}^{\sqrt{3}} = \frac{2\sqrt{3}}{3}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\pi\sqrt{3}}{9}.$$

So the answer is (A).

19. Let $f(x)$ be a cubic polynomial with leading coefficient 1 and a root at $x = 0$. If $\int_0^1 f(x)\,dx = 1$ then what is the sum of all possible values of $f(\frac{2}{3})$?

Solution. Let $f(x) = x^3 + mx^2 + nx$ for real numbers m and n. We have from the integral equation that $1/4 + m/3 + n/2 = 1$, which implies $m/3 + n/2 = 3/4$. We are asked to find the possible values of $8/27 + 4m/9 + 2n/3$. This expression is equal to $8/27 + (4/3)(m/3 + n/2) = 8/27 + (4/3)(3/4) = 8/27 + 1 = 35/27$. (D)

20. Compute $\displaystyle\int_0^{\pi/4} \ln(1+\tan x)\ dx$.

Solution. Let $x = \pi/4 - u$, so that $dx = -du$. Then $\tan(\pi/4 - u) = (1 - \tan u)/(1 + \tan u)$ by the tangent addition identity. Thus, the integral after this substitution becomes

$$\int_{\pi/4}^{0} -\ln\left(1 + \frac{1-\tan u}{1+\tan u}\right) du = \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan u}\right) du = \int_0^{\pi/4} (\ln 2 - \ln(1+\tan u))\ du.$$

Therefore, we conclude that

$$\int_0^{\pi/4} \ln(1+\tan x)\,dx = \int_0^{\pi/4} (\ln 2 - \ln(1+\tan x))\,dx = \int_0^{\pi/4} \ln 2\,dx - \int_0^{\pi/4} \ln(1+\tan x))\,dx.$$

Let the value of the original integral be I and add it to both sides of the above equation. Then we have that $2I = \int_0^{\pi/4} \ln 2\,dx$ so $I = (\pi \ln 2)/8$. (A)

21. If $\lim_{x\to 0} \dfrac{\int_0^x f(t)\,dt}{x^2} = 4$, then what is the value of $f(0) + f'(0)$?

Solution. Applying L'Hôspital's rule we see that this limit becomes $\lim_{x\to 0} f(x)/(2x)$. If this limit is to equal a finite value (which it is), then we must have that $f(0) = 0$. We thus can apply L'Hôspital again to get that $\lim_{x\to 0} f'(x)/2 = 4$, so $f'(0) = 8$ and the answer is 8. (C)

22. Compute $\displaystyle\int_{1/2}^{2} \frac{x^4-1}{x^5+x}\,dx$.

Solution. The bounds suggest that the substitution $x = 1/u$ and $dx = -1/u^2\,du$ is a good try. When this is done, the integral becomes

$$\int_2^{1/2} \frac{1/u^4-1}{1/u^5+1/u} \cdot \frac{-1}{u^2}\,du = \int_{1/2}^{2} \frac{1/u^4-1}{1/u^3+u}\,du = \int_{1/2}^{2} \frac{1-u^4}{u+u^5}\,du.$$

Therefore we conlcude that

$$\int_{1/2}^{2} \frac{x^4-1}{x^5+x}\,dx = \int_{1/2}^{2} -\frac{x^4-1}{x^5+x}\,dx.$$

Because these two integrals are but the negations of each other, then the value of this integral must be 0. (A)

23. What is the smallest possible real value n for which $\displaystyle\int_0^1 \frac{\arctan x}{x^n}\,dx$ diverges?

Solution. On the interval $(0,1)$, the most significant term in the Maclaurin expansion of $\arctan x$ is x, as the higher order terms will be small when compared. $\int_0^1 1/x^p\,dx$ blows up when $p \geq 1$ and converges to a real value otherwise. Thus, because $\arctan x$ has most significant term x, we must have $n \geq 2$ for this integral to diverge. Thus, the smallest possible value that results in divergence is 2. (C)

24. Compute $\displaystyle\int_0^{\infty} \left(\frac{x+1}{x^2+1}\right)^2 e^{-x}\,dx$.

Solution. The denominator suggests that this integrand could be the result of a derivative taken of a function with denominator x^2+1. If the numerator is $f(x)$, then by the quotient rule, we must have $(x^2+1)f'(x) - 2xf(x) = (x+1)^2e^{-x}$. Upon inspection, we see that $f(x) = -e^{-x}$ is indeed a solution. Thus, the integral becomes

$$\lim_{b\to\infty} \left.\frac{-e^{-x}}{x^2+1}\right|_0^b = \lim_{b\to\infty} \left(\frac{-e^{-b}}{b^2+1} + 1\right) = 1.$$

So the answer is (D).

25. Compute $\int_0^1 x\left\lfloor \frac{1}{x}\right\rfloor\, dx$ where $\lfloor x\rfloor$ is the greatest integer less than or equal to x.

Solution. We can rewrite this integral as

$$\sum_{n=1}^{\infty}\int_{1/(n+1)}^{1/n} nx\ dx$$

by considering the separate values for which $\lfloor 1/x\rfloor$ is constant. Evaluating, we get this is

$$\begin{aligned}\sum_{n=1}^{\infty}\int_{1/(n+1)}^{1/n} nx\ dx &= \sum_{n=1}^{\infty}\left.\frac{nx^2}{2}\right|_{1/(n+1)}^{1/n}\\ &= \sum_{n=1}^{\infty}\left(\frac{1}{2n}-\frac{n}{2(n+1)^2}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{2n}-\frac{1}{2(n+1)}+\frac{1}{2(n+1)^2}\right).\end{aligned}$$

The sum of the first two terms can be isolated and summed as a telescoping series with sum $1/2$. The last term is half the sum of the reciprocals of the squares of all natural numbers but 1, which is $(\pi^2/6-1)/2$. The sum of these is $1/2+(\pi^2/6-1)/2 = 1/2+\pi^2/12-1/2 = \pi^2/12$. (E)

26. Evaluate $\int_0^{\pi/2}\frac{1}{1+\tan^{2022}(x)}\ dx$.

Solution. Recall that $\tan(\pi/2-\theta) = \cot(\theta) = 1/\tan(\theta)$. We use the substitutiuon $u = \pi/2-x$ and we get

$$\begin{aligned}I = \int_0^{\pi/2}\frac{1}{1+\tan^{2022}(x)}\ dx &= \int_0^{\pi/2}\frac{1}{1+\tan^{2022}(\pi/2-u)}\ du\\ &= \int_0^{\pi/2}\frac{1}{1+\tan^{-2022}(x)}\ dx = \int_0^{\pi/2}\frac{\tan^{2022}(x)}{1+\tan^{2022}(x)}\ dx.\end{aligned}$$

Adding the first and last integral gives $2I = \int_0^{\pi/2} 1\ dx = \pi/2$ so the final answer is $\pi/4$. (C)

27. Let $I = \int_6^{18}\arcsin\left(\sqrt{\frac{x}{x+6}}\right)\ dx$. Then I can be written in the form $a\pi - b\sqrt{c}+d$, where $a, b, c, d \in \mathbb{N}$ and c is squarefree (i.e. not divisible by the square of any prime). Find $a+b+c+d$.

Solution. Note that

$$\int \arcsin\left(\sqrt{\frac{x}{x+6}}\right)\ dx = \int \arctan\left(\sqrt{\frac{x}{6}}\right)\ dx.$$

Using integration by parts, let $u = \arctan(\sqrt{x/6})$ and $dv = dx$. Then v is actually x plus a constant; we choose a helpful constant. Let $v = x+6$ and

$$du = \frac{1}{1+x/6}\cdot\frac{1}{2\sqrt{x/6}}\cdot\frac{1}{6} = \frac{\sqrt{6}}{2\sqrt{x}(x+6)}.$$

Hence, we have

$$\begin{aligned}\int \arctan\left(\sqrt{\frac{x}{6}}\right)\ dx &= (x+6)\arctan\left(\sqrt{\frac{x}{6}}\right) - \int \frac{(x+6)\sqrt{6}}{2\sqrt{x}(x+6)}\ dx.\\ &= (x+6)\arctan\left(\sqrt{\frac{x}{6}}\right) - \frac{\sqrt{6}}{2}\int \frac{1}{\sqrt{x}}\ dx\\ &= (x+6)\arctan\left(\sqrt{\frac{x}{6}}\right) - \frac{\sqrt{6}}{2}\cdot 2\sqrt{x} + C\\ &= (x+6)\arctan\left(\sqrt{\frac{x}{6}}\right) - \sqrt{6x} + C.\end{aligned}$$

Plugging in the bounds then gives $5\pi - 6\sqrt{3} + 6$. This gives $a+b+c+d = 5+6+3+6 = 20$. (D)

28. Evaluate $\displaystyle\int_0^{\pi/2} \frac{\sin(2021x)}{\sin(x)}\ dx$.

Solution. Let $I_n = \int_0^{\pi/2} \sin((2n+1)x)/\sin(x)\ dx$. Now, consider the difference $I_n - I_{n-1}$:

$$I_n - I_{n-1} = \int_0^{\pi/2} \frac{\sin((2n+1)x) - \sin((2n-1)x)}{\sin(x)}\ dx.$$

We use an identity to rewrite the difference of the sines. We have

$$\begin{aligned}&\sin((2n+1)x) - \sin((2n-1)x)\\ &\qquad = \sin(2nx)\cos(x) + \sin(x)\cos(2nx) - (\sin(2nx)\cos(x) - \sin(x)\cos(2nx))\\ &\qquad = 2\cos(2nx)\sin(x).\end{aligned}$$

Thus,

$$I_n - I_{n-1} = \int_0^{\pi/2} \frac{2\cos(2nx)\sin(x)}{\sin(x)}\ dx = \int_0^{\pi/2} 2\cos(2nx)\ dx = 0.$$

Therefore, we see that I_n has the same value for every n. Hence, we calculate I_0:

$$I_{1010} = I_0 = \int_0^{\pi/2} \frac{\sin(x)}{\sin(x)}\ dx = \int_0^{\pi/2} 1\ dx = \frac{\pi}{2}.$$

So the answer is (C).

29. Approximate $\displaystyle\int_0^1 (8x^3 - 3x^2 + 2022x - 1000)\ dx$ using Simpson's rule with $n = 2022$ subdivisions.

Solution. Simpsons is exact for cubics, giving

$$\int_0^1 (8x^3 - 3x^2 + 2022x - 1000)\ dx = 2x^4 - x^3 + 1011x^2 - 1000x\Big|_0^1 = 12.$$

So the answer is (A).

30. Compute $\int_0^1 \int_0^1 \frac{1}{1-xy}\ dy\ dx$.

Solution. Because we are within the domain, we can write the integrand as a sum of geometric series:

$$\int_0^1 \int_0^1 \sum_{n=0}^{\infty} (xy)^n\ dy\ dx = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$$

after evaluating each separate layer of integrals (it is a symmetric region). This is the sum of the reciprocals of the squares of the natural numbers, which is $\pi^2/6$. (D)