1 Answers

1. A
2. A
3. C
4. B
5. D
6. A
7. C
8. B
9. B
10. E
11. B
12. A
13. B
14. B
15. B
16. B
17. D
18. A
19. D
20. A
21. C
22. A
23. C
24. D
25. E
26. C
27. D
28. C
29. A
30. D
2 Solutions

1. Compute \( \int_0^\pi \sin x \cos x \, dx \).

   Solution. Let \( u = \sin x, du = \cos x \, dx \) and the integral in \( x \) becomes an integral in \( u \) from 0 to 0 of \( u \, du \), which is 0. (A)

2. Compute \( \int_0^2 \frac{x}{2+x} \, dx \).

   Solution. Note that we can divide the fraction to yield
   \[
   \int_0^2 \frac{x}{2+x} \, dx = \int_0^2 \left(1 - \frac{2}{x+2}\right) \, dx = x - 2 \ln(x+2) \bigg|_0^2 = 2 - 2 \ln 4 + 2 \ln 2 = 2 - \ln 4.
   \]
   So the answer is (A).

3. Compute \( \int_0^{4\pi} |\sin 2x| \, dx \).

   Solution. Note that
   \[
   \int_0^{\pi/2} \sin 2x \, dx = -\frac{1}{2} \cos 2x \bigg|_0^{\pi/2} = \frac{1}{2} + \frac{1}{2} = 1.
   \]
   The interval from 0 to \( 4\pi \) can be broken down into 8 intervals of length \( \pi/2 \) where all values, and therefore all areas, are positive. Because \( \sin 2x \) is periodic, the value is \( 1 \cdot 8 = 8 \). (C)

4. Using a trapezoidal sum on 6 equal subintervals, estimate the value of \( \int_2^{20} (4x+1) \, dx \).

   Solution. A trapezoidal sum is exact for linear functions, so we can evaluate the integral directly to get \( 2 \cdot 20^2 + 20 - 2 \cdot 2^2 - 2 = 810 \). (B)

5. What is the total area bounded by the graph of \( f(x) = x^3 \) and its inverse \( f^{-1}(x) \)?

   Solution. Two separate regions (between \(-1\) and 0, and 0 and 1) are bounded, and by symmetry with respect to \( y = x \), the areas of each region are equal. The total bounded area is thus
   \[
   2 \int_0^1 (x^{1/3} - x^3) \, dx = 2 \left(\frac{3}{4} x^{4/3} - \frac{1}{4} x^4\right) \bigg|_0^1 = 2 \left(\frac{3}{4} - \frac{1}{4}\right) = 1.
   \]
   So the answer is (D).

6. Compute \( \lim_{n \to \infty} \sum_{i=0}^{n} \frac{i}{i^2 + n^2} \).

   Solution. Dividing each term by \( n^2 \), we can rewrite the summand as
   \[
   \frac{i/n^2}{i^2/n^2 + n^2/n^2} = \frac{i/n \cdot 1/n}{(i/n)^2 + 1} = \frac{1}{n} \cdot \frac{i/n}{(i/n)^2 + 1}.
   \]
By the Riemann definition of an integral, this sum is equal to
\[
\int_{0}^{1} \frac{x}{x^2 + 1} \, dx = \left. \frac{1}{2} \ln(x^2 + 1) \right|_{0}^{1} = \frac{\ln 2}{2}.
\]
So the answer is (A).

7. What is the length of the polar curve \( r = \theta^2 \) from \( \theta = 0 \) to \( \theta = 2 \)?

**Solution.** By the polar arc length definition, we integrate the square root of \( r^2 + (r')^2 \). Thus, we have that the length of this curve is
\[
\int_{0}^{2} \sqrt{\theta^4 + 4\theta^2} \, d\theta = \int_{0}^{2} \theta \sqrt{\theta^2 + 4} \, d\theta = \left. \frac{1}{3} (\theta^2 + 4)^{3/2} \right|_{0}^{2} = \frac{8^{3/2} - 4^{3/2}}{3} = \frac{16\sqrt{2} - 8}{3}.
\]
So the answer is (C).

8. If \( g(x) \) is an even function and \( \int_{\mathbb{R}} g(x) \, dx = 4 \) then compute the value of \( \int_{0}^{\infty} g(x) \, dx \).

**Solution.** Because the integral in question is over all positive real numbers and our function is even, this is simply equal to \( 4/2 = 2 \). (B)

9. \( e^x (\cos x - \sin x) \) is the derivative of which of the following?

**Solution.** The given function is the derivative of \( e^x \cos x \) through the product rule. (B)

10. Compute \( \int_{0}^{\infty} \sum_{n=0}^{\infty} (-x^2)^n \, dx \).

**Solution.** The inner summand evaluates to \( 1/(1 + x^2) \) by the sum of an infinite geometric series, provided \( 0 < x < 1 \). But because \( x \) goes from 0 to \( \infty \), this integral diverges. (E)

11. The function \( f(x) = kx(1 - x)^3 \) defines a probability density function on \([0, 1]\) for some real \( k \). Compute \( k \).

**Solution.** To solve for \( k \), we note that \( \int_{0}^{1} kx(1 - x)^3 \, dx = 1 \) in order for this to be a probability distribution. Hence, using integration by parts with \( u = x \) and \( dv = (1 - x)^3 \), we obtain
\[
\int_{0}^{1} kx(1 - x)^3 \, dx = 1
\]
\[
- \frac{kx}{4} (1 - x)^4 \bigg|_{0}^{1} - \frac{k}{20} (1 - x)^5 \bigg|_{0}^{1} = 1
\]
\[
k \frac{1}{20} = 1,
\]
so \( k = 20 \). (B)

12. Let \( R \) be the region bounded by the parametric equations \( x(t) = 2t \) and \( y(t) = t/(t^2 + 1) \) and the \( x \) axis over the interval \( t \in [0, 1] \). What is the area of \( R \)?
The area is given by $\int y \, dx$, which becomes $\int_0^1 2t/(t^2 + 1) \, dt$ by plugging in $y(t)$ and computing $dx = 2 \, dt$. This integral is easily solved through $u$-substitution and is equal to $\ln 2$. (A)

13. Compute $\int_0^\infty \frac{x^2}{(x^2 + 1)^2} \, dx$.

Solution. Let $x = \tan \theta$ so that $dx = \sec^2 \theta \, d\theta$. Then the integral becomes

$$\int_0^{\pi/2} \sin^2 \theta \, d\theta = \int_0^{\pi/2} \frac{1}{2} (1 - \cos 2\theta) \, d\theta = \frac{1}{4} (2\theta - \sin 2\theta) \bigg|_0^{\pi/2} = \frac{\pi}{4}.$$ 

So the answer is (B).

14. Compute $\int_1^\infty \frac{x^2 \ln^2 x}{x^2} \, dx$.

Solution. Let $u = \ln x$ so that $e^u \, du = dx$. The integral then becomes, using integration by parts,

$$\int_0^\infty u^2 e^{-u} \, du = \lim_{b \to \infty} -(u^2 + 2u + 2)e^{-u} \bigg|_0^b = 2.$$ 

So the answer is (B).

15. Compute $\int_0^{2\pi} \sin(\sin x - x) \, dx$.

Solution. Use sine addition on the integrand to rewrite it as $\sin(\sin x) \cos x - \cos(\sin x) \sin x$. Now we integrate each term. The integral of $\sin(\sin x) \cos x$ over $[0, 2\pi]$ is equal to 0, which can be seen through the $u$-substitution $u = \sin x$:

$$\int_0^{2\pi} \sin(\sin x) \cos x \, dx = -\cos(\sin x) \bigg|_0^{2\pi} = -\cos(0) + \cos(2\pi) = 0.$$ 

As for the second term, we use the $u$-substitution $x = 2\pi - u$ and the identity $\sin(2\pi - u) = -\sin u$:

$$\int_0^{2\pi} -\cos(\sin x) \sin x \, dx = \int_0^0 \cos(\sin(2\pi - u)) \sin(2\pi - u) \, du$$
$$= -\int_0^{2\pi} \cos(-\sin u) \sin u \, du = \int_0^{2\pi} \cos(\sin u) \sin u \, du.$$ 

Thus, this transformation takes $\cos(\sin x) \sin(x)$ to $-\cos(\sin x) \sin x$, so this integral must also be 0. Therefore, the entire integral is 0. (B)

16. Compute $\int_0^1 x \ln(1 - x) \, dx$.

Solution. We can write (on the interval $(0, 1)$)

$$\ln(1 - x) = -\sum_{i=1}^\infty \frac{x^i}{i}, \quad \text{so that} \quad x \ln(1 - x) = -\sum_{i=1}^\infty \frac{x^{i+1}}{i}.$$
Therefore, we can integrate the series:

\[
\int_0^1 x \ln(1-x) \, dx = - \int_0^1 \sum_{i=1}^{\infty} \frac{x^{i+1}}{i} \, dx = - \left( \sum_{i=1}^{\infty} \frac{x^{i+2}}{i(i+2)} \right) \bigg|_0^1 = - \sum_{i=1}^{\infty} \frac{1}{i(i+2)} = - \sum_{i=1}^{\infty} \frac{1}{2} \left( \frac{1}{i} - \frac{1}{i+2} \right)
\]

This series telescopes, leaving \(- (1/2)(1 + 1/2) = -3/4\). (B)

17. Compute \(\int_0^4 (x - 7)(x - 2)^5 \, dx\).

**Solution.** Let \(u = x - 2\). The integral becomes

\[
\int_{-2}^{2} (u - 5)u^5 \, du = \int_{-2}^{2} (u^6 - 5u^5) \, du.
\]

Because the integral of the odd function \(-5u^5\) is 0 over \([-2, 2]\), the integral is simply \(2 \cdot 2^7 / 7 = 2 \cdot 128 / 7 = 256 / 7\). (D)

18. Compute \(\int_0^1 \frac{1}{1 + x + x^2} \, dx\).

**Solution.** We can complete the square on the integrand to rewrite it as

\[
\frac{1}{(x + 1/2)^2 + 3/4} = \frac{4/3}{((2x + 1)/\sqrt{3})^2 + 1}.
\]

Let \(u = (2x + 1)/\sqrt{3}\). Then \(du = 2/\sqrt{3} \, dx\) to get that this integral is

\[
\frac{4\sqrt{3}}{6} \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{1}{1 + u^2} \, du = \frac{2\sqrt{3}}{3} \arctan u \bigg|_{1/\sqrt{3}}^{\sqrt{3}} = \frac{2\sqrt{3}}{3} \left( \frac{\pi}{3} - \frac{\pi}{6} \right) = \frac{\pi\sqrt{3}}{9}.
\]

So the answer is (A).

19. Let \(f(x)\) be a cubic polynomial with leading coefficient 1 and a root at \(x = 0\). If \(\int_0^1 f(x) \, dx = 1\) then what is the sum of all possible values of \(f(\frac{2}{3})\)?

**Solution.** Let \(f(x) = x^3 + mx^2 + nx\) for real numbers \(m\) and \(n\). We have from the integral equation that \(1/4 + m/3 + n/2 = 1\), which implies \(m/3 + n/2 = 3/4\). We are asked to find the possible values of \(8/27 + 4m/9 + 2n/3\). This expression is equal to \(8/27 + (4/3)(m/3 + n/2) = 8/27 + (4/3)(3/4) = 8/27 + 1 = 35/27\). (D)

20. Compute \(\int_0^{\pi/4} \ln(1 + \tan x) \, dx\).

**Solution.** Let \(x = \pi/4 - u\), so that \(dx = -du\). Then \(\tan(\pi/4 - u) = (1 - \tan u)/(1 + \tan u)\) by the tangent addition identity. Thus, the integral after this substitution becomes

\[
\int_{\pi/4}^0 -\ln \left( \frac{1 - \tan u}{1 + \tan u} \right) \, du = \int_0^{\pi/4} \ln \left( \frac{2}{1 + \tan u} \right) \, du = \int_0^{\pi/4} (\ln 2 - \ln(1 + \tan u)) \, du.
\]
Therefore, we conclude that
\[
\int_0^{\pi/4} \ln(1 + \tan x) \, dx = \int_0^{\pi/4} (\ln 2 - \ln(1 + \tan x)) \, dx = \int_0^{\pi/4} \ln 2 \, dx - \int_0^{\pi/4} \ln(1 + \tan x) \, dx.
\]
Let the value of the original integral be \( I \) and add it to both sides of the above equation. Then we have that \( 2I = \int_0^{\pi/4} \ln 2 \, dx \) so \( I = (\pi \ln 2)/8 \). (A)

21. If \( \lim_{x \to 0} \frac{f(t)}{x^2} = 4 \), then what is the value of \( f(0) + f'(0) \)?

**Solution.** Applying L'Hôpital's rule we see that this limit becomes \( \lim_{x \to 0} \frac{f(x)}{2x} \). If this limit is to equal a finite value (which it is), then we must have that \( f(0) = 0 \). We thus can apply L'Hôpital again to get that \( \lim_{x \to 0} \frac{f'(x)}{x} = 4 \), so \( f'(0) = 8 \) and the answer is 8. (C)

22. Compute \( \int_{1/2}^{2} \frac{x^4 - 1}{x^5 + x} \, dx \).

**Solution.** The bounds suggest that the substitution \( x = 1/u \) and \( dx = -1/u^2 \, du \) is a good try. When this is done, the integral becomes
\[
\int_{1/2}^{2} \frac{1/u^4 - 1}{1/u^5 + 1/u} \cdot \frac{-1}{u^2} \, du = \int_{1/2}^{2} \frac{1/u^4 - 1}{1/u^5 + u} \, du = \int_{1/2}^{2} \frac{1 - u^4}{u + u^5} \, du.
\]
Therefore we conclude that
\[
\int_{1/2}^{2} \frac{x^4 - 1}{x^5 + x} \, dx = \int_{1/2}^{2} -\frac{x^4 - 1}{x^5 + x} \, dx.
\]
Because these two integrals are but the negations of each other, then the value of this integral must be 0. (A)

23. What is the smallest possible real value \( n \) for which \( \int_0^1 \frac{\arctan x}{x^n} \, dx \) diverges?

**Solution.** On the interval \((0,1)\), the most significant term in the Maclaurin expansion of \( \arctan x \) is \( x \), as the higher order terms will be small when compared. \( \int_0^1 x^p \, dx \) blows up when \( p \geq 1 \) and converges to a real value otherwise. Thus, because \( \arctan x \) has most significant term \( x \), we must have \( n \geq 2 \) for this integral to diverge. Thus, the smallest possible value that results in divergence is 2. (C)

24. Compute \( \int_0^\infty \left( \frac{x + 1}{x^2 + 1} \right)^2 e^{-x} \, dx \).

**Solution.** The denominator suggests that this integrand could be the result of a derivative taken of a function with denominator \( x^2 + 1 \). If the numerator is \( f(x) \), then by the quotient rule, we must have \((x^2 + 1)f'(x) - 2xf(x) = (x + 1)^2 e^{-x} \). Upon inspection, we see that \( f(x) = -e^{-x} \) is indeed a solution. Thus, the integral becomes
\[
\lim_{b \to \infty} \left. -e^{-x} \right|_0^b = \lim_{b \to \infty} \left( -e^{-b} - \frac{e^{-b}}{b^2 + 1} + 1 \right) = 1.
\]
So the answer is (D).
25. Compute \( \int_0^1 x \left\lfloor \frac{1}{x} \right\rfloor \, dx \) where \( \lfloor x \rfloor \) is the greatest integer less than or equal to \( x \).

\textbf{Solution.} We can rewrite this integral as

\[ \sum_{n=1}^{\infty} \int_{1/(n+1)}^{1/n} nx \, dx \]

by considering the separate values for which \( \lfloor 1/x \rfloor \) is constant. Evaluating, we get this is

\[ \sum_{n=1}^{\infty} \int_{1/(n+1)}^{1/n} nx \, dx = \sum_{n=1}^{\infty} \left( \frac{1}{2n} - \frac{n}{2(n+1)^2} \right) = \sum_{n=1}^{\infty} \left( \frac{1}{2n} - \frac{1}{2(n+1)} + \frac{1}{2(n+1)^2} \right) \cdot \]

The sum of the first two terms can be isolated and summed as a telescoping series with sum \( 1/2 \). The last term is half the sum of the reciprocals of the squares of all natural numbers but 1, which is \( (\pi^2/6 - 1)/2 \). The sum of these is \( 1/2 + (\pi^2/6 - 1)/2 = 1/2 + \pi^2/12 - 1/2 = \pi^2/12 \). (E)

26. Evaluate \( \int_0^{\pi/2} \frac{1}{1 + \tan^{2022}(x)} \, dx \).

\textbf{Solution.} Recall that \( \tan(\pi/2 - \theta) = \cot(\theta) = 1/\tan(\theta) \). We use the substitution \( u = \pi/2 - x \) and we get

\[ I = \int_0^{\pi/2} \frac{1}{1 + \tan^{2022}(x)} \, dx = \int_0^{\pi/2} \frac{1}{1 + \tan^{2022}(\pi/2 - u)} \, du \]

\[ = \int_0^{\pi/2} \frac{1}{1 + \tan^{-2022}(x)} \, dx = \int_0^{\pi/2} \frac{\tan^{2022}(x)}{1 + \tan^{2022}(x)} \, dx. \]

Adding the first and last integral gives \( 2I = \int_0^{\pi/2} 1 \, dx = \pi/2 \) so the final answer is \( \pi/4 \). (C)

27. Let \( I = \int_6^{18} \arcsin\left(\sqrt{x/(x+6)}\right) \, dx \). Then \( I \) can be written in the form \( a\pi - b\sqrt{c} + d \), where \( a, b, c, d \in \mathbb{N} \) and \( c \) is squarefree (i.e. not divisible by the square of any prime). Find \( a+b+c+d \).

\textbf{Solution.} Note that

\[ \int \arcsin\left(\sqrt{\frac{x}{x+6}}\right) \, dx = \int \arctan\left(\sqrt{\frac{x}{6}}\right) \, dx. \]

Using integration by parts, let \( u = \arctan(\sqrt{x/6}) \) and \( dv = dx \). Then \( v \) is actually \( x \) plus a constant; we choose a helpful constant. Let \( v = x + 6 \) and

\[ du = \frac{1}{1 + x/6} \cdot \frac{1}{2\sqrt{x/6}} \cdot \frac{1}{6} = \frac{\sqrt{6}}{2\sqrt{x}(x+6)}. \]
Hence, we have
\[ \int \arctan \left( \sqrt{\frac{x}{6}} \right) \, dx = (x + 6) \arctan \left( \sqrt{\frac{x}{6}} \right) - \int \frac{(x + 6)\sqrt{6}}{2\sqrt{x}(x + 6)} \, dx. \]
\[ = (x + 6) \arctan \left( \sqrt{\frac{x}{6}} \right) - \frac{\sqrt{6}}{2} \int \frac{1}{\sqrt{x}} \, dx \]
\[ = (x + 6) \arctan \left( \sqrt{\frac{x}{6}} \right) - \frac{\sqrt{6}}{2} \cdot 2\sqrt{x} + C \]
\[ = (x + 6) \arctan \left( \sqrt{\frac{x}{6}} \right) - \sqrt{6}x + C. \]

Plugging in the bounds then gives \( 5\pi - 6\sqrt{3} + 6 \). This gives \( a + b + c + d = 5 + 6 + 3 + 6 = 20 \).

(D)

28. Evaluate \( \int_{0}^{\pi/2} \frac{\sin(2021x)}{\sin(x)} \, dx \).

**Solution.** Let \( I_n = \int_{0}^{\pi/2} \frac{\sin((2n + 1)x)}{\sin(x)} \, dx \). Now, consider the difference \( I_n - I_{n-1} \):
\[ I_n - I_{n-1} = \int_{0}^{\pi/2} \frac{\sin((2n + 1)x) - \sin((2n - 1)x)}{\sin(x)} \, dx. \]
We use an identity to rewrite the difference of the sines. We have
\[ \sin((2n + 1)x) - \sin((2n - 1)x) = \sin(2nx)\cos(x) + \sin(x)\cos(2nx) - (\sin(2nx)\cos(x) - \sin(x)\cos(2nx)) = 2\cos(2nx)\sin(x). \]
Thus,
\[ I_n - I_{n-1} = \int_{0}^{\pi/2} \frac{2\cos(2nx)\sin(x)}{\sin(x)} \, dx = \int_{0}^{\pi/2} 2\cos(2nx) \, dx = 0. \]
Therefore, we see that \( I_n \) has the same value for every \( n \). Hence, we calculate \( I_0 \):
\[ I_{1010} = I_0 = \int_{0}^{\pi/2} \frac{\sin(x)}{\sin(x)} \, dx = \int_{0}^{\pi/2} 1 \, dx = \frac{\pi}{2}. \]
So the answer is (C).

29. Approximate \( \int_{0}^{1} (8x^3 - 3x^2 + 2022x - 1000) \, dx \) using Simpson’s rule with \( n = 2022 \) subdivisions.

**Solution.** Simpsons is exact for cubics, giving
\[ \int_{0}^{1} (8x^3 - 3x^2 + 2022x - 1000) \, dx = 2x^4 - x^3 + 1011x^2 - 1000x \bigg|_{0}^{1} = 12. \]
So the answer is (A).
30. Compute $\int_0^1 \int_0^1 \frac{1}{1-xy} \, dy \, dx$.

Solution. Because we are within the domain, we can write the integrand as a sum of geometric series:

$$\int_0^1 \int_0^1 \sum_{n=0}^{\infty} (xy)^n \, dy \, dx = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$$

after evaluating each separate layer of integrals (it is a symmetric region). This is the sum of the reciprocals of the squares of the natural numbers, which is $\frac{\pi^2}{6}$. (D)