Instructions For This Test: All functions and variables are real-valued unless otherwise specified, and $i = \sqrt{-1}$. "NOTA" stands for "None Of These Answers is correct." Good luck and have fun!

1. Find the eccentricity of the conic $2025y^2 + 2025!^{2025}(y - 2025) + (y - x) \cdot 2025(x + y) + (\sqrt{2025}x + 2025!)^2 = 0$.

(A) 0 **(B)** 1 **(C)** 2025 **(D)** $\frac{1}{2025}$ **(E)** NOTA

2. Lucas is given two three-dimensional vectors $\vec{u}$ and $\vec{v}$ and tasked with computing $|\vec{u} + \vec{v}|$. He mistakenly finds $|\vec{u}| + |\vec{v}|$ instead, but still gets the correct answer. Which of the following must be true?

(A) $|\vec{u} - \vec{v}| = 0$ **(B)** $|\vec{u} \cdot \vec{v}| = 0$ **(C)** $|\vec{u} \times \vec{v}| = 0$ **(D)** $|\vec{u} + \vec{v}| = 0$ **(E)** NOTA

3. Which of the following equations describes the conic of eccentricity 2 with a focus at $(4, 0)$ and corresponding directrix $x = 1$?

(A) $x^2 - y^2 + 4x - 14 = 0$ **(C)** $2x^2 + 2y^2 - 17x + 33 = 0$ **(E)** NOTA
(B) $x^2 + y^2 - 10x + 18 = 0$ **(D)** $x^2 + 2y^2 - 14x + 31 = 0$

4. The point $(\sin(t), \cos(2t))$ lies on a conic in the xy-plane for all values of t. Find the length of the latus rectum of this conic.

(A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$ **(C)** 1 **(D)** 2 **(E)** NOTA

5. If the acute angle between the planes $9x + 6y - 2z = 7$ and $-8x + 4y + z = 11$ is θ, what is $\cos(\theta)$?

(A) $\frac{50}{99}$ **(B)** $-\frac{50}{99}$ **(C)** $\frac{50}{77}$ **(D)** $-\frac{50}{77}$ **(E)** NOTA

6. Which of the following lines parameterized by t passes through the points $(2, -3, -1)$ and $(4, -5, 4)$?

(A) $l(t) = \langle 2, -2, 5\rangle t - \langle 2, -3, -1\rangle$ **(C)** $l(t) = \langle 2, -3, -1\rangle t + \langle 2, -2, -5\rangle$ **(E)** NOTA
(B) $l(t) = \langle 2, -2, 5\rangle t + \langle 0, -1, -6\rangle$ **(D)** $l(t) = \langle 2, -3, -1\rangle t + \langle 4, -5, 4\rangle$

7. Find the area of the convex polygon in the xy-plane with vertices $(-4, -3)$, $(-7, 2)$, $(1, 1)$, $(-5, 3)$, and $(2, -3)$.

(A) $\frac{21}{2}$ **(B)** $\frac{39}{2}$ **(C)** 33 **(D)** $\frac{71}{2}$ **(E)** NOTA

8. Grogu and Mando are hiding away on the planet Sorgan. Grogu's hut can be modeled by the pyramid bounded by the plane $2x + 3y + 6z = 12$ and the coordinate planes. What is the volume of Grogu's hut?

(A) 16 **(B)** 48 **(C)** 36 **(D)** 8 **(E)** NOTA

9. Grogu's metal ball can be modeled by a sphere inscribed inside the hut from the previous question. What is the radius of Grogu's ball?

(A) $\frac{12}{7}$ **(B)** $\frac{6}{7}$ **(C)** $\frac{2}{3}$ **(D)** 3 **(E)** NOTA

10. Which of the following is a directrix of the polar graph $r = \frac{8}{2+\cos(\theta)}$?

(A) $r = -\frac{4}{\cos(\theta)}$ **(B)** $r = -\frac{8}{\cos(\theta)}$ **(C)** $r = \frac{4}{\cos(\theta)}$ **(D)** $r = \frac{8}{\cos(\theta)}$ **(E)** NOTA

11. Find the area enclosed by the polar graph $r = \frac{8}{2+\cos(\theta)}$.

(A) 16π **(B)** $\frac{32\sqrt{3}}{3}\pi$ **(C)** $\frac{128\sqrt{3}}{9}\pi$ **(D)** $\frac{128}{9}\pi$ **(E)** NOTA

12. Meghan is standing in the corner $(0,0)$ of the room in the xy-plane bounded by $x = 0$, $x = 1$, $y = 0$, and $y = 1$. She unwisely shoots a laser with slope $\frac{3}{4}$ that reflects off of the room's walls and reflects directly backwards if it hits a corner. How many units will the laser travel before returning to Meghan?

(A) 5 **(B)** 10 **(C)** 15 **(D)** 20 **(E)** NOTA

13. Identify the non-degenerate conic defined by the equation $2x^2 - 6xy + 5y^2 + 7x - 4y - 12 = 0$.

(A) Parabola **(B)** Circle **(C)** Hyperbola **(D)** Ellipse **(E)** NOTA

14. Anagh is at the origin and wants to get to his real analysis class at the point $(5,3,2)$. If he can only take unit steps in the positive x, y, and z directions, how many distinct paths can he take to class?

(A) 252 **(B)** 540 **(C)** 2520 **(D)** 5400 **(E)** NOTA

15. A basis $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ of $\mathbb{R}^3$ is a list of vectors such that any vector $\vec{v} = \langle x, y, z\rangle$ can be expressed as a unique linear combination of the vectors in $\mathcal{B}$ (That is, $\vec{v} = \alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n$ for a unique choice of coefficients $\alpha_1, \ldots, \alpha_n$). Which of the following is a basis of $\mathbb{R}^3$?

(A) $\big(\langle 1,2,3\rangle, \langle 1,2,4\rangle, \langle 1,3,4\rangle\big)$
(B) $\big(\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle\big)$
(C) $\big(\langle 1,1,0\rangle, \langle 1,0,-1\rangle, \langle 0,1,0\rangle, \langle 0,-1,1\rangle\big)$
(D) $\big(\langle 8,2,1\rangle, \langle 11,5,10\rangle, \langle 10,4,7\rangle\big)$
(E) NOTA

16. The graph of $|z - 2 + i| = r|z - 1 - i|$ encloses a region in the Argand plane for $0 < r < 1$. Find the area of this region in terms of r.

(A) $\frac{5r^2}{1-r^2}\pi$ **(B)** $\frac{5r^4}{(1-r^2)^2}\pi$ **(C)** $\frac{5r^2}{(1-r^2)^2}\pi$ **(D)** $\frac{5}{1-r^2}\pi$ **(E)** NOTA

17. Which of the following best describes the graph of $(x^2 + y^2 + ax)^2 - a^2(x^2 + y^2) = 0$ for an arbitrary constant $a > 0$?

(A) Cardioid **(B)** Cycloid **(C)** Lemniscate **(D)** Rose curve **(E)** NOTA

18. The complex number $a + bi$ is transformed to $a' + b'i$ by the following matrix multiplication.

$$\begin{bmatrix} 2 & 3i \\ -i & 4 \end{bmatrix} \begin{bmatrix} a \\ bi \end{bmatrix} = \begin{bmatrix} a' \\ b'i \end{bmatrix}$$

If the hexagon formed by the roots of $x^6 = 1$ in the Argand plane undergoes this transformation, what is the area of the resulting shape?

(A) $9\sqrt{3}$ **(B)** $\frac{33\sqrt{3}}{2}$ **(C)** $\frac{3\sqrt{3}}{2}$ **(D)** $12\sqrt{3}$ **(E)** NOTA

19. What is the center of the ellipse $5x^2 + 4xy + 2y^2 - 22x - 4y + 23 = 0$?

(A) $(\frac{22}{5}, 2)$ **(B)** $(-3, 2)$ **(C)** $(\frac{11}{5}, 1)$ **(D)** $(3, -2)$ **(E)** NOTA

20. Which of the following points is a focus of the ellipse $5x^2 + 4xy + 2y^2 = 6$?

(A) $(2, 1)$ **(B)** $(-1, 2)$ **(C)** $(3, \frac{3}{2})$ **(D)** $(-\frac{3}{2}, 3)$ **(E)** NOTA

21. Find the equation of the plane tangent to the ellipsoid $4x^2 + 4y^2 + z^2 = 36$ at the point $(2, 1, 4)$.

(A) $x + 2y + 2z = 12$ **(C)** $2x + y + 4z = 21$ **(E)** NOTA
(B) $2x + y + z = 9$ **(D)** $2x + 4y + z = 12$

22. How many of the following are degenerate cases of the graph of $ax^2 + bxy + cy^2 + dx + ey + f = 0$, but not the intersection of a plane and a (non-degenerate) double napped cone?

I. Empty graph II. Point III. Two parallel lines
IV. Line segment V. Line VI. Two intersecting lines

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** NOTA

Use the following information for the next 3 questions.

The director circle of an ellipse or hyperbola is the locus of all points where any two perpendicular tangents to the ellipse or hyperbola intersect.

23. What is the radius of the director circle of the ellipse $9x^2 + 16y^2 = 144$?

(A) 4 **(B)** 5 **(C)** $2\sqrt{7}$ **(D)** 7 **(E)** NOTA

24. Which of the following conics has a degenerate director circle?

(A) $3y^2 - 4x^2 = 1$ **(B)** $x^2 + y^2 = 1$ **(C)** $3x^2 - 8y^2 = 1$ **(D)** $6x^2 - 5y^2 = 1$ **(E)** NOTA

25. The circle $x^2 + y^2 = 1$ is the director circle of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{c^2} - \frac{y^2}{d^2} = 1$. If $ac = 1$, and the product of the eccentricities of both conics is 1, what is a^2?

(A) $\frac{3-\sqrt{5}}{2}$ **(B)** 1 **(C)** $\frac{2-\sqrt{2}}{2}$ **(D)** $\frac{4}{9}$ **(E)** NOTA

26. Complex numbers a and b satisfy $a+b=1$ and $|a|+|b|=2$. The set of all possible values of $a-b$ encloses a region in the Argand plane. Find the area of this region.

(A) 4π **(B)** $\frac{\sqrt{3}}{2}\pi$ **(C)** π **(D)** $2\sqrt{3}\pi$ **(E)** NOTA

27. If the point on the line $l(t)=\langle 2,2,3\rangle t+\langle 6,3,1\rangle$ which is closest to the line $m(t)=\langle 5,4,7\rangle t+\langle 9,6,5\rangle$ is $\langle a,b,c\rangle$, what is $a+b+c$?

(A) $\frac{196}{9}$ **(B)** $\frac{775+7\sqrt{10}}{39}$ **(C)** $\frac{900+16\sqrt{17}}{117}$ **(D)** $\frac{65}{3}$ **(E)** NOTA

28. The conic $4x^2+8xy-11y^2-7x+3y+1=0$ is rotated about the origin to produce the conic $a'x^2+c'y^2+d'x+e'y+1=0$. Given that $a'<c'$, what is a'?

(A) -11 **(B)** -12 **(C)** -13 **(D)** -14 **(E)** NOTA

29. Allen is exploring the Argand plane, beginning at the point 0. In the n^{th} step, he travels to the midpoint of his current position and the point $e^{\frac{n\pi}{3}i}$. Where is Allen standing after 12 steps?

(A) $\frac{(2^{12}-1)\sqrt{3}i}{3\cdot 2^{12}}$ **(B)** $\frac{(2^{12}-1)(3+\sqrt{3}i)}{3\cdot 2^{13}}$ **(C)** $\frac{(2^{12}-1)(3-\sqrt{3}i)}{3\cdot 2^{13}}$ **(D)** $\frac{(1-2^{12})\sqrt{3}i}{3\cdot 2^{12}}$ **(E)** NOTA

30. Vishakha is at the point $(2,0,4)$ and shines a light towards the sphere $x^2+y^2+(z-2)^2=1$, casting a shadow onto the xy-plane. Find the area of this shadow.

(A) $\frac{16\sqrt{21}}{9}\pi$ **(B)** $\frac{16\sqrt{2}}{3}\pi$ **(C)** $\frac{56\sqrt{2}}{9}\pi$ **(D)** $\frac{112}{9}\pi$ **(E)** NOTA