Answer Key:

1. B
2. C
3. A
4. B
5. A
6. B
7. D
8. D
9. C
10. D
11. C
12. B
13. D
14. C
15. A
16. C
17. A
18. E
19. D
20. B
21. B
22. C
23. B
24. D
25. C
26. D
27. D
28. B
29. C
30. A

Solutions:

1. $\boxed{\textbf{B}}$: Expanding and simplifying,

$$2025y^2 + 2025!^{2025}y - 2025!^{2025} \cdot 2025 + 2025y^2 - 2025x^2 + 2025x^2 + 2 \cdot \sqrt{2025}x \cdot 2025! + 2025!^2 = 0$$

$$\implies 4050y^2 + 2025!^{2025}y - 2025!^{2025} \cdot 2025 + 2025!^2 = -2 \cdot \sqrt{2025} \cdot 2025!x$$

This is a parabola, and thus has eccentricity $\boxed{1}$.

2. $\boxed{\textbf{C}}$: Lucas's answer is correct, so $|\vec{u}+\vec{v}| = |\vec{u}| + |\vec{v}| = \sqrt{|\vec{u}|^2 + |\vec{v}|^2 + 2|\vec{u}||\vec{v}|}$. For $|\vec{u}|, |\vec{v}| \neq 0$, applying law of cosines where θ is the angle between $\vec{u}$ and $\vec{v}$ gives $|\vec{u}+\vec{v}| = \sqrt{|\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos(\pi - \theta)}$. Thus, $-\cos(\pi-\theta) = \cos(\theta) = 1$, so $\sin(\theta) = 0$ and $|\vec{u}||\vec{v}|\sin(\theta) = \boxed{|\vec{u} \times \vec{v}| = 0}$. Note that this is also true when $|\vec{u}| = 0$ or $|\vec{v}| = 0$.

3. $\boxed{\textbf{A}}$: By definition of eccentricity, the distance from any point on this conic to the focus $(4,0)$ is 2 times the distance from the point to the directrix $x = 1$.

$$\sqrt{(x-4)^2 + y^2} = 2|x-1| \implies x^2 - 8x + 16 + y^2 = 2x^2 - 4x + 2 \implies \boxed{x^2 - y^2 + 4x - 14 = 0}$$

4. $\boxed{\textbf{B}}$: Applying the cosine double angle identity, $y = \cos(2t) = 1 - 2\sin^2(t) = 1 - 2x^2$. This is a parabola, whose latus rectum has length $4a$ where $(x-h)^2 = \pm 4a(y-k)$. We have $x^2 = -\frac{1}{2}(y-1) \implies 4a = \boxed{\frac{1}{2}}$.

5. $\boxed{\textbf{A}}$: The angle between two planes is equal to the angle θ between the normal vectors $\vec{n}_1$ and $\vec{n}_2$ of the planes.

$$\vec{n}_1 = \langle 9, 6, -2 \rangle, \ \ \vec{n}_2 = \langle -8, 4, 1 \rangle$$

$$\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1||\vec{n}_2|\cos(\theta) \implies \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \cos(\theta) \implies \frac{9 \cdot (-8) + 6 \cdot 4 - 2 \cdot 1}{\sqrt{9^2 + 6^2 + (-2)^2}\sqrt{(-8)^2 + 4^2 + 1^2}} = -\frac{50}{99} = \cos(\theta)$$

Since this is negative, θ is the obtuse angle between the planes. The cosine of the acute angle between the planes is $\cos(\pi - \theta) = -\cos(\theta) = \boxed{\frac{50}{99}}$.

6. $\boxed{\textbf{B}}$: The direction vector is $\langle 4, -5, 4 \rangle - \langle 2, -3, -1 \rangle = \langle 2, -2, 5 \rangle$ and the point $\langle 2, -3, -1 \rangle$ is on the line. Thus,

$$l(t) = \langle 2, -2, 5 \rangle t + \langle 2, -3, -1 \rangle$$

Since t is an arbitrary parameter, we can substitute $t - 1$ instead.

$$l(t) = \langle 2, -2, 5 \rangle (t-1) + \langle 2, -3, -1 \rangle = \langle 2, -2, 5 \rangle t - \langle 2, -2, 5 \rangle + \langle 2, -3, -1 \rangle \implies \boxed{l(t) = \langle 2, -2, 5 \rangle t + \langle 0, -1, -6 \rangle}$$

7. $\boxed{\textbf{D}}$: The vertices of the polygon, in counterclockwise order, are $(1,1)$, $(-5,3)$, $(-7,2)$, $(-4,-3)$, and $(2,-3)$. Applying shoelace formula, the area of the polygon is:

$$\frac{\left|\left(1 \cdot 3 - 5 \cdot 2 - 7 \cdot (-3) - 4 \cdot (-3) + 2 \cdot 1\right) - \left(1 \cdot (-5) + 3 \cdot (-7) + 2 \cdot (-4) - 3 \cdot 2 - 3 \cdot 1\right)\right|}{2} = \boxed{\frac{71}{2}}$$

8. $\boxed{\textbf{D}}$: The plane intersects the x-axis where $2x + 3 \cdot 0 + 6 \cdot 0 = 12 \implies x = 6$. Similarly, the y and z-intecepts are 4 and 2 respectively. Thus, the base of the pyramid is a right triangle in the xy-plane with legs 6 and 4 and area 12, and the height of the pyramid is 2. Therefore, the pyramid has volume $\frac{1}{3} \cdot 12 \cdot 2 = \boxed{8}$.

9. $\boxed{\text{C}}$: For a sphere of radius r to be tangent to the coordinate planes, its center must be (r, r, r). It must also be a distance of r from the plane $2x + 3y + 6z = 12$. Plugging this into the point to plane distance formula,

$$\frac{|2r + 3r + 6r - 12|}{\sqrt{2^2 + 3^2 + 6^2}} = r \implies \frac{|11r - 12|}{7} = r$$

Thus, $11r - 12 = 7r \implies r = 3$ or $11r - 12 = -7r \implies r = \frac{2}{3}$. The first case corresponds to the circle externally tangent to the four planes, so the inscribed circle has radius $r = \boxed{\frac{2}{3}}$.

10. $\boxed{\text{D}}$: Multiplying by $2 + \cos(\theta)$ and converting to cartesian coordinates,

$$r = \frac{8}{2 + \cos(\theta)} \implies 2r + r\cos(\theta) = 8 \implies 2\sqrt{x^2 + y^2} + x = 8 \implies \sqrt{x^2 + y^2} = \frac{1}{2}(8 - x)$$

Thus, the distance from (x, y) to the origin is $\frac{1}{2}$ times the distance to the line $x = 8$, so $x = 8$ is a directrix. Converting back to polar, $r\cos(\theta) = 8 \implies \boxed{r = \frac{8}{\cos(\theta)}}$.

11. $\boxed{\text{C}}$: Multiplying by $2 + \cos(\theta)$ and converting to cartesian coordinates,

$$r = \frac{8}{2 + \cos(\theta)} \implies 2r + r\cos(\theta) = 8 \implies 2\sqrt{x^2 + y^2} + x = 8 \implies 2\sqrt{x^2 + y^2} = 8 - x$$

Squaring and completing the square,

$$4x^2 + 4y^2 = x^2 - 16x + 64 \implies 3x^2 + 16x + 4y^2 - 64 = 0$$

$$\implies 3\left(x + \frac{8}{3}\right)^2 + 4y^2 = \frac{256}{3} \implies \frac{\left(x + \frac{8}{3}\right)^2}{\frac{256}{9}} + \frac{y^2}{\frac{64}{3}} = 1$$

Thus, the polar graph is an ellipse with area $\frac{16}{3} \cdot \frac{8}{\sqrt{3}} \cdot \pi = \boxed{\frac{128\sqrt{3}}{9}\pi}$.

12. $\boxed{\text{B}}$: Each reflection off of a wall can be "unfolded" by mirroring the laser's path across the wall, resulting in the following diagram where the laser travels in a straight line. The first two reflections are displayed in color.

The dotted vertices are mirrors of Meghan's location. The laser first intersects one of these vertices at the point $(8, 6)$ after traveling a distance of $\sqrt{8^2 + 6^2} = \boxed{10}$.

13. $\boxed{\text{D}}$: The discriminant of this conic is $b^2 - 4ac = (-6)^2 - 4 \cdot 2 \cdot 5 = -4 < 0$, so it is an $\boxed{\text{Ellipse}}$.

14. $\boxed{\text{C}}$: Anagh has to take a total of 10 steps, 3 of which being in the y direction and 2 of the remaining 7 being in the z direction. Thus, the number of possible paths is $\binom{10}{3}\binom{7}{2} = \boxed{2520}$.

15. $\boxed{\text{A}}$: Consider an arbitrary $\vec{v} = \langle x, y, z \rangle$. For any list, if we can determine unique values of $\alpha_1, \ldots, \alpha_n$ in terms of x, y and z, then the list is a basis of $\mathbb{R}^3$.

For answer choice A, $\langle x, y, z \rangle = \alpha_1 \langle 1, 2, 3 \rangle + \alpha_2 \langle 1, 2, 4 \rangle + \alpha_3 \langle 1, 3, 4 \rangle$, and we have the following system of equations.

$$\alpha_1 + \alpha_2 + \alpha_3 = x$$

$$2\alpha_1 + 2\alpha_2 + 3\alpha_3 = y$$
$$3\alpha_1 + 4\alpha_2 + 4\alpha_3 = z$$

Solving gives the unique values $\alpha_1 = 4x - z$, $\alpha_2 = -x - y + z$, and $\alpha_3 = y - 2x$. Thus, $\boxed{(\langle 1,2,3\rangle, \langle 1,2,4\rangle, \langle 1,3,4\rangle)}$ is a basis of $\mathbb{R}^3$. Note that answer choices B and C will result in 3 equations for 4 variables and therefore not yield any unique values. Solving a similar system for answer choice D does not yield unique values of $\alpha_1, \ldots, \alpha_n$ either.

16. $\boxed{\text{C}}$: Note that translations and rotations of the graph do not change the enclosed area. Since the distance between $2 - i$ and $1 + i$ is $\sqrt{(2-1)^2 + (-1-1)^2} = \sqrt{5}$, the enclosed region has the same area as the region enclosed by $|z| = r|z - \sqrt{5}|$. Converting to cartesian coordinates,

$$\sqrt{x^2+y^2} = r\sqrt{(x-\sqrt{5})^2 + y^2} \Longrightarrow x^2 + y^2 = r^2(x^2 - 2\sqrt{5}x + 5 + y^2)$$

$$\Longrightarrow (1-r^2)x^2 + 2\sqrt{5}r^2x + (1-r^2)y^2 = 5r^2 \Longrightarrow x^2 + \frac{2\sqrt{5}r^2}{1-r^2}x + y^2 = \frac{5r^2}{1-r^2}$$

$$\Longrightarrow \left(x + \frac{\sqrt{5}r^2}{1-r^2}\right)^2 + y^2 = \frac{5r^2}{1-r^2} + \frac{5r^4}{(1-r^2)^2} = \frac{5r^2}{(1-r^2)^2}$$

Thus, the region is a circle with area $\boxed{\frac{5r^2}{(1-r^2)^2}\pi}$.

17. $\boxed{\text{A}}$: Converting to polar coordinates,

$$(r^2 + ar\cos(\theta))^2 - a^2r^2 = 0 \Longrightarrow (r^2 + ar\cos(\theta))^2 = (ar)^2$$

$$\Longrightarrow r^2 + ar\cos(\theta) = ar \Longrightarrow r + a\cos(\theta) = a \Longrightarrow r = a(1 - \cos(\theta))$$

This is the polar equation of a $\boxed{\text{Cardioid}}$.

18. $\boxed{\text{E}}$: Expanding the given matrix multiplication:

$$\begin{bmatrix} 2 & 3i \\ -i & 4 \end{bmatrix} \begin{bmatrix} a \\ bi \end{bmatrix} = \begin{bmatrix} a' \\ b'i \end{bmatrix} = \begin{bmatrix} 2a - 3b \\ (-a + 4b)i \end{bmatrix}$$

This is equivalent to the following transformation in the xy-plane:

$$\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2x - 3y \\ -x + 4y \end{bmatrix}$$

This matrix has determinant $2 \cdot 4 - (-1) \cdot (-3) = 5$ and therefore scales the area of a transformed region by a factor of 5. The hexagon formed by the roots of $x^6 = 1$ is made up of 6 equilateral triangles of side length 1 and thus has area $6 \cdot \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{2}$. Thus, the area of the transformed hexagon is $5 \cdot \frac{3\sqrt{3}}{2} = \boxed{\frac{15\sqrt{3}}{2}}$.

19. $\boxed{\text{D}}$: Note that if a point (x, y) is on the ellipse $5x^2 + 4xy + 2y^2 + f = 0$, then $(-x, -y)$ is also on the ellipse, so the ellipse has center $(0, 0)$. Translating, $5(x-h)^2 + 4(x-h)(y-k) + 2(y-k)^2 + f = 0$ has center (h, k). Expanding,

$$5x^2 + 4xy + 2y^2 + (-10h - 4k)x + (-4h - 4k)y + (f + 5h^2 + 4hk + 2k^2) = 0$$

Matching the coefficients of x and y, we have the following system of equations.

$$-10h - 4k = -22 \qquad -4h - 4k = -4$$

$$\Longrightarrow -10h - 4k - (-4h - 4k) = -6h = -22 - (-4) = -18 \Longrightarrow h = 3,\ k = -2$$

Thus, the center of the ellipse is $\boxed{(3, -2)}$.

20. $\boxed{\textbf{B}}$: If the rotation angle of the ellipse is θ, then we have $\cot(2\theta) = \frac{A-C}{B} = \frac{3}{4}$, so $\cos(2\theta) = \frac{3}{5}$. Applying half angle formulas, $\cos(\theta) = \frac{2}{\sqrt{5}}$ and $\sin(\theta) = \frac{1}{\sqrt{5}}$, so $\tan(\theta) = \frac{1}{2}$. Since the ellipse is centered on the origin, the axes of the ellipse lie on $y = \frac{1}{2}x$ and $y = -2x$. Plugging these in,

$$5x^2 + 4x \cdot \frac{1}{2}x + 2\left(\frac{1}{2}x\right)^2 = 6 \implies \frac{15}{2}x^2 = 6 \implies x = \pm\frac{2}{\sqrt{5}},\ y = \pm\frac{1}{\sqrt{5}}$$

$$5x^2 + 4x(-2x) + 2(-2x)^2 = 6 \implies 5x^2 = 6 \implies x = \pm\frac{\sqrt{6}}{\sqrt{5}},\ y = \mp\frac{2\sqrt{6}}{\sqrt{5}}$$

The semi-axis along $y = \frac{1}{2}x$ has length $\sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = 1$ and the semi-axis along $y = -2x$ has length $\sqrt{\left(\frac{\sqrt{6}}{\sqrt{5}}\right)^2 + \left(\frac{2\sqrt{6}}{\sqrt{5}}\right)^2} = \sqrt{6}$. Thus, the foci (x, y) are on $y = -2x$ with focal radius $\sqrt{\left(\sqrt{6}\right)^2 - 1^2} = \sqrt{5}$, so $\sqrt{x^2 + (-2x)^2} = \sqrt{5x^2} = \sqrt{5} \implies x = \pm 1$. Therefore, the foci are $(1, -2)$ and $\boxed{(-1, 2)}$.

21. $\boxed{\textbf{B}}$: Note that stretching the ellipse and the plane equally preserves tangency. Stretching the ellipse $4x^2 + 4y^2 + z^2 = 36$ and the point $(2, 1, 4)$ by a factor of $\frac{1}{2}$ in the z direction results in the sphere $x^2 + y^2 + z^2 = 9$ and the point $(2, 1, 2)$. The tangent plane at this point has normal vector $\langle 2, 1, 2\rangle$ and thus has equation $2x + y + 2z = 9$. Stretching this plane back out by a factor of 2 in the z direction results in the plane $\boxed{2x + y + z = 9}$.

22. $\boxed{\textbf{C}}$: The graph of $ax^2 + bxy + cy^2 + dx + ey + f = 0$ can be any of the following (examples given).

I. Empty graph ($x^2 + 1 = 0$)
II. Point ($x^2 + y^2 = 0$)
III. Two parallel lines ($x(x - 1) = 0$)
V. Line ($x = 0$)
VI. Two intersecting lines ($xy = 0$)

The intersection of a plane and double napped cone is a degenerate conic when the plane passes through the vertex of the cones. This can be any of the following.

II. Point (plane is horizontal)
V. Line (plane has same slope as cone)
VI. Two intersecting lines (plane is vertical)

Thus, only I and III are degenerate cases of $ax^2 + bxy + cy^2 + dx + ey + f = 0$ but not the intersection of a plane and double napped cone, totaling to $\boxed{2}$ cases.

23. $\boxed{\textbf{B}}$: The line tangent to the ellipse at the vertex $(4, 0)$ is $x = 4$ and the line tangent to the ellipse at the covertex $(0, 3)$ is $y = 3$. These tangents are perpendicular, so their intersection $(4, 3)$ must be on the director circle. Since the director circle must be concentric to the ellipse, it is centered at the origin and has radius $\sqrt{(4-0)^2 + (3-0)^2} = \boxed{5}$.

24. $\boxed{\text{D}}$: Consider the hyperbola $6x^2 - 5y^2 = 1$ and its asymptotes of slope $\pm\sqrt{\frac{6}{5}}$ graphed below.

The slope m of any tangent to this hyperbola must be in the range $\left(-\infty, -\sqrt{\frac{6}{5}}\right) \cup \left(\sqrt{\frac{6}{5}}, \infty\right)$. Thus, the slope $-\frac{1}{m}$ of a line perpendicular to any tangent is in the range $\left(-\sqrt{\frac{5}{6}}, \sqrt{\frac{5}{6}}\right)$. Since this is not in the range of slopes of tangents to the hyperbola, there cannot be two perpendicular tangents, so the director circle of $\boxed{6x^2 - 5y^2 = 1}$ is degenerate.

25. $\boxed{\text{C}}$: Applying the same procedure as in question 23, the ellipse has perpendicular tangents $y = b$ and $x = a$. Thus, the radius of its director circle is $r = \sqrt{a^2 + b^2} = 1$, so $b = \sqrt{1 - a^2}$.
The point $(1, 0)$ is on the director circle of the hyperbola and corresponds to the perpendicular tangents $y = \pm(x - 1)$ as graphed below.

For these to be tangent to the hyperbola, there must be only one solution to $y = x - 1$ and $\frac{x^2}{c^2} - \frac{y^2}{d^2} = 1$. Plugging in,

$$\frac{x^2}{c^2} - \frac{(x-1)^2}{d^2} = 1 \implies d^2x^2 - c^2(x-1)^2 = c^2d^2$$

$$\implies d^2x^2 - c^2x^2 + 2c^2x - c^2 = c^2d^2 \implies (d^2 - c^2)x^2 + 2c^2x - c^2(1 + d^2) = 0$$

For this quadratic to only have one solution, the discriminant $b^2 - 4ac$ must equal 0.

$$(2c^2)^2 - 4 \cdot (d^2 - c^2) \cdot (-c^2)(1 + d^2) = 0 \implies c^2 + (d^2 - c^2)(1 + d^2) = 0$$

$$\implies c^2 + d^2 - c^2 + d^4 - d^2c^2 = 0 \implies 1 + d^2 - c^2 = 0 \implies d = \sqrt{c^2 - 1}$$

The eccentricity of the hyperbola is $\frac{\sqrt{c^2+d^2}}{c} = \frac{\sqrt{2c^2-1}}{c} = \sqrt{2 - \frac{1}{c^2}} = \sqrt{2 - a^2}$ after substituting $c = \frac{1}{a}$.

If the semi-major axis of the ellipse is a, then it has eccentricity $\frac{\sqrt{a^2-b^2}}{a} = \frac{\sqrt{2a^2-1}}{a} = \sqrt{2 - \frac{1}{a^2}}$. The product of the eccentricities of both conics is 1, so we have the following.

$$\sqrt{2 - \frac{1}{a^2}}\sqrt{2 - a^2} = 1 \implies \left(2 - \frac{1}{a^2}\right)(2 - a^2) = 1 \implies 5 - 2a^2 - \frac{2}{a^2} = 1$$

$$\implies 2a^4 - 4a^2 + 2 = 0 \implies 2(a^2 - 1)^2 = 0 \implies a^2 = 1$$

However, this makes $b = \sqrt{1 - a^2} = 0$, resulting in a degenerate ellipse (and division by 0). Thus, the semi-major axis must be b, so its eccentricity is $\frac{\sqrt{b^2-a^2}}{b} = \frac{\sqrt{1-2a^2}}{\sqrt{1-a^2}}$ and we have the following.

$$\frac{\sqrt{1-2a^2}}{\sqrt{1-a^2}}\sqrt{2 - a^2} = 1 \implies \sqrt{1 - 2a^2}\sqrt{2 - a^2} = \sqrt{1 - a^2} \implies (1 - 2a^2)(2 - a^2) = (1 - a^2)$$

$$\implies 2a^4 - 5a^2 + 2 = 1 - a^2 \implies 2a^4 - 4a^2 + 1 = 0 \implies a^2 = \frac{4 \pm \sqrt{8}}{4} = \frac{2 \pm \sqrt{2}}{2}$$

Since $b = \sqrt{1 - a^2}$, we know a^2 must be less than 1, so $a^2 = \boxed{\frac{2 - \sqrt{2}}{2}}$.

26. $\boxed{\text{D}}$: Treating a and b as vectors and adding them tip-to-tail, we have the following diagram.

Note that the conditions $a+b=1$ and $|a+b|=2$ mean that a lies on the ellipse centered at $(\frac{1}{2},0)$ with major axis 2 and focal radius $\frac{1}{2}$. Every point $a-b$ is twice as far from the focus $(1,0)$ as a is, essentially scaling the ellipse by a factor of 2 about the focus and thus scaling its area by a factor of 4. Our original ellipse has semi-major axis 1 and semi-minor axis $\sqrt{1^2-(\frac{1}{2})^2}=\frac{\sqrt{3}}{2}$, so its area is $\frac{\sqrt{3}}{2}\pi$. Thus, the desired area is $4\cdot\frac{\sqrt{3}}{2}\pi=\boxed{2\sqrt{3}\pi}$.

27. $\boxed{\text{D}}$: The normal vector $\vec{n}$ to both lines is the cross product of the direction vectors $\langle 2,2,3\rangle$ and $\langle 5,4,7\rangle$.

$$\vec{n}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 2 & 3\\ 5 & 4 & 7\end{vmatrix}=\langle 2,1,-2\rangle$$

Taking $p=l(0)=\langle 6,3,1\rangle$ and $q=m(0)=\langle 9,6,5\rangle$, we have the vector $\vec{pq}=\langle 3,3,4\rangle$ from a point on l to a point on m. Applying skew line distance formula, the minimum distance between l and m is the following.

$$\frac{|\vec{pq}\cdot\vec{n}|}{|\vec{n}|}=\frac{|3(2)+3(1)+4(-2)|}{\sqrt{2^2+1^2+(-2)^2}}=\frac{1}{3}$$

Taking $p=l(t)=\langle 2t+6,2t+3,3t+1\rangle$ and $q=m(0)=\langle 9,6,5\rangle$, we have the vector $\vec{pq}=\langle -2t+3,-2t+3,-3t+4\rangle$ from $l(t)$ to a point on m. Applying point to line distance formula with $\vec{m}=\langle 5,4,7\rangle$, the distance between $l(t)$ and m is $\frac{|\vec{pq}\times\vec{m}|}{|\vec{m}|}$, and we have the following.

$$\vec{pq}\times\vec{m}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -2t+3 & -2t+3 & -3t+4\\ 5 & 4 & 7\end{vmatrix}=\langle -2t+5,-t-1,2t-3\rangle$$

$$|\vec{pq}\times\vec{m}|=\sqrt{(-2t+5)^2+(-t-1)^2+(2t-3)^2}=\sqrt{9t^2-30t+35}$$

$$\frac{|\vec{pq}\times\vec{m}|}{|\vec{m}|}=\frac{\sqrt{9t^2-30t+35}}{\sqrt{5^2+4^2+7^2}}=\frac{\sqrt{9t^2-30t+35}}{\sqrt{90}}=\frac{1}{3}$$

$$\implies\sqrt{9t^2-30t+35}=\sqrt{10}\implies 9t^2-30t+25=0\implies(3t-5)^2=0\implies t=\frac{5}{3}$$

Plugging in, $l(\frac{5}{3})=\langle 2\cdot\frac{5}{3}+6,2\cdot\frac{5}{3}+3,3\cdot\frac{5}{3}+1\rangle=\langle\frac{28}{3},\frac{19}{3},6\rangle$. The desired sum is $\frac{28}{3}+\frac{19}{3}+6=\boxed{\dfrac{65}{3}}$.

28. $\boxed{\text{B}}$: Under rotation about the origin where $f=f'=1$, we have the invariants $a+c=a'+c'$ and $b^2-4ac=-4a'c'$.

$$(a+c)^2+b^2-4ac=(a'+c')^2-4a'c'\implies(a-c)^2+b^2=(a'-c')^2$$

$$a'<c'\implies\sqrt{(a-c)^2+b^2}=c'-a'$$

$$\frac{a+c-\sqrt{(a-c)^2+b^2}}{2}=\frac{a'+c'-(c'-a')}{2}=a'=\frac{4-11-\sqrt{(4+11)^2+8^2}}{2}=\boxed{-12}$$

29. $\boxed{\text{C}}$: Denote $z(n)$ as Allen's position after n steps, so $z(n)=\frac{1}{2}z(n-1)+\frac{1}{2}e^{\frac{n\pi}{3}i}$ where $z(0)=0$.

$$z(1)=\frac{1}{2}(0)+\frac{1}{2}\cdot e^{\frac{\pi}{3}i}=\frac{1}{2}e^{\frac{\pi}{3}i}$$

$$z(2)=\frac{1}{2}\left(\frac{1}{2}e^{\frac{\pi}{3}i}\right)+\frac{1}{2}\cdot e^{\frac{2\pi}{3}i}=\frac{1}{4}e^{\frac{\pi}{3}i}+\frac{1}{2}e^{\frac{2\pi}{3}i}$$

$$\vdots$$

$$z(12) = \frac{1}{2^{12}}e^{\frac{\pi}{3}i} + \frac{1}{2^{11}}e^{\frac{2\pi}{3}i} + \cdots + \frac{1}{2}e^{\frac{12\pi}{3}i} = \sum_{n=1}^{12} \frac{1}{2^{13-n}}e^{\frac{n\pi}{3}i}$$

Evaluating as a geometric series and plugging in $2e^{\frac{\pi}{3}i} = 1 + \sqrt{3}i$,

$$\frac{1}{2^{13}}\sum_{n=1}^{12}(2e^{\frac{\pi}{3}i})^n = \frac{1}{2^{13}}\frac{(2e^{\frac{\pi}{3}i})^{13} - 2e^{\frac{\pi}{3}i}}{2e^{\frac{\pi}{3}i} - 1} = \frac{2e^{\frac{\pi}{3}i}((2e^{\frac{\pi}{3}i})^{12} - 1)}{2^{13}(2e^{\frac{\pi}{3}i} - 1)}$$

$$= \frac{(2^{12} - 1)(1 + \sqrt{3}i)}{2^{13}\sqrt{3}i} = \boxed{\frac{(2^{12} - 1)(3 - \sqrt{3}i)}{3 \cdot 2^{13}}}$$

30. $\boxed{\textbf{A}}$: The region blocked from the light by the sphere is a cone, which will form an elliptical shadow when intersected by the xy-plane. Taking the xz-plane as a cross section, we have the following diagram.

The vertices of the ellipse are A and C, and the center of the ellipse is H, the midpoint of AC.

We have $RF = 1$ and $VR = 2\sqrt{2}$, so $VF = \sqrt{(2\sqrt{2})^2 - 1^2} = \sqrt{7}$.

Note that $\triangle BCD$, $\triangle BCE$, $\triangle BDE$, $\triangle AGB$, and $\triangle HIB$ are isoceles right triangles. We have $BV = 4\sqrt{2}$. Let $BE = EC = ED = x$ so that $EV = 4\sqrt{2} - x$. By similar triangles,

$$\frac{VF}{RF} = \frac{VE}{CE} \implies \sqrt{7} = \frac{4\sqrt{2} - x}{x} \implies 1 + \sqrt{7} = \frac{4\sqrt{2}}{x} \implies x = \frac{4\sqrt{2}}{1 + \sqrt{7}} = \frac{2\sqrt{14} - 2\sqrt{2}}{3}$$

Let $AG = GB = y$ so that $VG = 4\sqrt{2} + y$. Note that $\angle AVG = \angle CVE$, so by similar triangles,

$$\frac{VF}{RF} = \frac{VG}{AG} \implies \sqrt{7} = \frac{4\sqrt{2} + y}{y} \implies \sqrt{7} - 1 = \frac{4\sqrt{2}}{y} \implies y = \frac{4\sqrt{2}}{\sqrt{7} - 1} = \frac{2\sqrt{14} + 2\sqrt{2}}{3}$$

Thus, $AB = AG\sqrt{2} = \frac{4\sqrt{7}+4}{3}$ and $BC = BE\sqrt{2} = \frac{4\sqrt{7}-4}{3}$, so the major axis $AC = \frac{4\sqrt{7}+4}{3} + \frac{4\sqrt{7}-4}{3} = \frac{8\sqrt{7}}{3}$ and the semi-major axis $AH = \frac{4\sqrt{7}}{3}$

Furthermore, $HB = AB - AH = \frac{4\sqrt{7}+4}{3} - \frac{4\sqrt{7}}{3} = \frac{4}{3}$, and $HI = \frac{HB}{\sqrt{2}} = \frac{2\sqrt{2}}{3}$. Since H is the midpoint of AC, JI is the median of the trapezoid $AGED$ and has length $\frac{AG+DE}{2} = \frac{2\sqrt{14}-2\sqrt{2}}{6} + \frac{2\sqrt{14}+2\sqrt{2}}{6} = \frac{2\sqrt{14}}{3}$. Taking a cross section perpendicular to the axis of the cone, we have the following diagram.

The semi-minor axis of the ellipse is HL and has length $\sqrt{IL^2 - IH^2} = \sqrt{\left(\frac{2\sqrt{14}}{3}\right)^2 - \left(\frac{2\sqrt{2}}{3}\right)^2} = \frac{4\sqrt{3}}{3}$. Thus, the area of the ellipse is $\frac{4\sqrt{7}}{3} \cdot \frac{4\sqrt{3}}{3} \cdot \pi = \boxed{\frac{16\sqrt{21}}{9}\pi}$.