0. $\mathbf{17}$

1. $\mathbf{25\pi}$

2. $\mathbf{8\pi}$

3. $\mathbf{2}$

4. $\mathbf{18}$

5. $\mathbf{24}$

6. $\mathbf{-82}$

7. $\mathbf{5\sqrt{2}}$

8. $\mathbf{20}$

9. $\mathbf{\frac{1}{3}}$

10. $\mathbf{7}$

0: 17

$\begin{vmatrix} \log_4 9 & \log_4 3 \\ \log_3 8 & \log_3 512 \end{vmatrix} = \begin{vmatrix} \log_4 3^2 & \log_4 3 \\ \log_3 2^3 & \log_3 2^9 \end{vmatrix} = \begin{vmatrix} \log_2 3 & \frac{1}{2}\log_2 3 \\ 3\log_3 2 & 9\log_3 2 \end{vmatrix}$ by Change of Base.
This determinant is $\log_2 3 \cdot 9\log_3 2 - 3\log_3 2 \cdot \frac{1}{2}\log_2 3 = 9 - \frac{3}{2} = \frac{15}{2}$.
Thus, $m + n = 15 + 2 = 17$.

1: 25π

One approach is to consider the perpendicular bisectors of the chords, whose intersection is the center of the circle. The first chord has slope $\frac{3+1}{0+2} = 2$ and midpoint $\left(\frac{-2+0}{2}, \frac{-1+3}{2}\right) = (-1,1)$.
Its perpendicular bisector has equation $y = -\frac{1}{2}(x+1) + 1$. The second chord has slope $\frac{4+1}{3+2} = 1$ and midpoint $\left(\frac{-2+3}{2}, \frac{-1+4}{2}\right) = \left(\frac{1}{2}, \frac{3}{2}\right)$. Its perpendicular bisector has equation $y = -1\left(x - \frac{1}{2}\right) + \frac{3}{2}$.
Solve $-\frac{1}{2}(x+1) + 1 = -1\left(x - \frac{1}{2}\right) + \frac{3}{2}$ to find $x = 3$ and substitute to find $y = -1$. The radius of the circle is the distance between $(3,-1)$ and $(-2,-1)$ which equals 5. The area of the circle is then $\pi \cdot 5^2 = 25\pi$.

2: 8π

Use the sum to product identities $\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$ and $\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$. The numerator $\cos 3x - \cos 11x = -2\sin(7x)\sin(-4x)$ and the denominator $\sin 9x + \sin 5x = 2\sin 7x \sin 2x$. Thus, $\frac{\cos 3x - \cos 11x}{\sin 9x + \sin 5x} = \frac{-2\sin(7x)\sin(-4x)}{2\sin(7x)\cos(2x)} = -\frac{\sin(-4x)}{\cos(2x)}$. Since sine is odd, this becomes $\frac{\sin(4x)}{\cos(2x)}$.
By a double angle identity, this simplifies to $\frac{2\sin(2x)\cos(2x)}{\cos(2x)} = 2\sin(2x)$. Now our equation becomes $(2\sin 2x)^2 = 2$ or $\sin^2 2x = \frac{1}{2}$. Square root to get $\sin 2x = \pm\frac{\sqrt{2}}{2}$. Let $u = 2x$ so that $\sin u = \pm\frac{\sqrt{2}}{2}$ whose solutions are $u = \frac{\pi}{4} + \pi k, \frac{3\pi}{4} + \pi k$ where k is any integer. Since $x = \frac{u}{2}$, $x = \frac{\pi}{8} + \frac{\pi}{2}k, \frac{3\pi}{8} + \frac{\pi}{2}k$. On $[0, 2\pi)$, the solutions are $x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$. This forms an arithmetic series of 8 terms with first term $a_1 = \frac{\pi}{8}$ and last term $a_8 = \frac{15\pi}{8}$. The sum is $\frac{a_1 + a_8}{2} \cdot 8 = \frac{\frac{\pi}{8} + \frac{15\pi}{8}}{2} \cdot 8 = 8\pi$. Note: Each of these angles is in the domain of the original equation.

3. 2

By the Change of Base Property, notice $\log_4 u = \frac{\log_2 u}{\log_2 4} = \frac{\log_2 u}{2} = \frac{1}{2}\log_2 u = \log_2 \sqrt{u}$.
Also by Change of Base, $\log_8 u = \frac{\log_2 u}{\log_2 8} = \frac{\log_2 u}{3} = \frac{1}{3}\log_2 u = \log_2 \sqrt[3]{u}$.

The system now becomes $\begin{cases} \log_2 x + \log_2 \sqrt{y} + \log_2 \sqrt[3]{z} = \frac{1}{6} \\ \log_2 \sqrt{x} + \log_2 \sqrt[3]{y} + \log_2 z = -\frac{1}{2} \\ \log_2 \sqrt[3]{x} + \log_2 y + \log_2 \sqrt{z} = -\frac{1}{2} \end{cases}$. Use the sum to product property of logs to write the system as $\begin{cases} \log_2(x\sqrt{y}\,\sqrt[3]{z}) = \frac{1}{6} \\ \log_2(\sqrt{x}\,\sqrt[3]{y}z) = -\frac{1}{2} \\ \log_2(\sqrt[3]{x}\sqrt{y}z) = \frac{13}{6} \end{cases}$. Exponentiate to write $\begin{cases} x\sqrt{y}\sqrt[3]{z} = 2^{\frac{1}{6}} \\ \sqrt{x}\sqrt[3]{y}z = 2^{-\frac{1}{2}} \\ \sqrt[3]{x}y\sqrt{z} = 2^{\frac{13}{6}} \end{cases}$.

Now raise both sides of each equation to the power 6: $\begin{cases} x^6y^2z = 2 \\ x^3y^3z^6 = 2^{-3} \\ x^2y^6z^3 = 2^{13} \end{cases}$. Multiply all three equations to get $x^{11}y^{11}z^{11} = 2^{11}$, showing $xyz = 2$.

4. 18

Let the smallest angle measure θ so the largest angle measures 2θ. By Law of Sines, $\frac{\sin 2\theta}{\sin\theta} = \frac{4}{3}$. By a double angle identity, $\frac{2\sin\theta\cos\theta}{\sin\theta} = 2\cos\theta$. Thus, $\cos\theta = \frac{2}{3}$. By the Pythagorean identity, $\sin\theta = \sqrt{1-\left(\frac{2}{3}\right)^2} = \frac{\sqrt{5}}{3}$. Thus, $\sin(2\theta) = 2\left(\frac{\sqrt{5}}{3}\right)\left(\frac{2}{3}\right) = \frac{4\sqrt{5}}{9}$. Therefore $m+n+p = 4+5+9 = 18$.

5. 24

Rearrange the first equation as $8x^2 - 16x + 18y^2 + 36y = 118$. Now complete the square as $8(x-1)^2 + 18(y+1)^2 = 144$ or $\frac{(x-1)^2}{18} + \frac{(y+1)^2}{8} = 1$, which is an ellipse. Rearrange the second equation as $2x^2 - 4x - 3y^2 - 6y = 7$. Complete the square as $2(x-1)^2 - 3(y+1)^2 = 6$ so $\frac{(x-1)^2}{3} - \frac{(y+1)^2}{2} = 1$, which is a hyperbola with the same center $(1,-1)$. Multiply this equation by $\frac{1}{4}$ and add to the ellipse equation: $\frac{5(x-1)^2}{36} = \frac{5}{4}$ so $(x-1)^2 = 9$ thus $x = 1 \pm 3$ so $x=-2, x=4$. Substitute to find y which yields the four vertices $(-2,1), (4,1)$, $(4,-3), (-2,-3)$. These points form a rectangle with base 6 and height 4 whose area is 24.

6. -82

Denote the roots as $\frac{q}{r} < q < rq$ since the roots are in geometric progression. The product $\frac{q}{r}\cdot q\cdot rq = q^3 = 64 \rightarrow q = 4$. The sum $\frac{4}{r} + 4 + 4r = -6$ so $\frac{4}{r} + 4r + 10 = 0$.
Simplify to $4r^2 + 10r + 4 = 0 = 2(2r+1)(r+2) = 0$ so $r = -\frac{1}{2}$ or $r = -2$. The roots are $-2, 4, -8$ so $f(x) = (x+2)(x-4)(x+8) = x^3 + 6x^2 - 24x - 64$. Therefore, $a+b+c = 6 - 24 - 64 = -82$.

7. $5\sqrt{2}$

Let $z = x + iy$. Then $x^2 + (y+1)^2 = 53$ and $(x+3)^2 + y^2 = 101$. Therefore $x^2 + y^2 + 2y = 52$ and $x^2 + 6x + y^2 = 92$. Subtract the two equations to get $3x - y = 20$. Substitute $y = 3x - 20$ into the first equation: $x^2 + (3x - 19)^2 = 53$ so $5x^2 - 57x + 154 = 0$. Factor as $(5x - 22)(x - 7) = 0$ so $x = \frac{22}{5}$ or $x = 7$. Then $y = -\frac{34}{5}$ or $y = 1$, respectively. Thus, $|z| = \sqrt{\left(\frac{22}{5}\right)^2 + \left(-\frac{34}{5}\right)^2} = \sqrt{65.6}$ or $|z| = \sqrt{50}$. The minimum value of $|z| = \sqrt{50} = 5\sqrt{2}$.

8. 20

Use sine's double angle identity $\sin 2\theta = 2\sin\theta\cos\theta$ to write $r = 8\sin 2\theta\cos 6\theta + 4\sin 4\theta$. Use the product-sum identity $\cos A \sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$ to write $r = 4(\sin 8\theta - \sin 4\theta) + 4\sin 4\theta$. This simplifies to $r = 4\sin 8\theta$. The polar graph has $2 \cdot 8 = 16$ petals since the coefficient of θ is even so $m = 16$. The maximum distance from the origin (and the length of each petal) is $n = 4$ due to the coefficient of 4 in the equation. Thus, $m + n = 20$.

9. $\frac{1}{3}$

Let X be the number of times Alfred spins red, out of 6 total independent spins. Then X is binomally distributed. $P(X=2) = \binom{6}{2}p^2(1-p)^4 = 15p^2(1-p)^4$ and $P(X=4) = \binom{6}{4}p^4(1-p)^2 = 15p^4(1-p)^2$. Thus, $15p^2(1-p)^4 = 4 \cdot 15p^4(1-p)^2$. Expand and simplify to $3p^2 + 2p - 1 = 0$ or $(3p-1)(p+1) = 0$. So $p = \frac{1}{3}$ since $p > 0$.

10. 7

The angle between $y = kx$ and $y = 2x$ must be congruent to the angle between $y = 2x$ and $y = x$. For any line with positive slope m through the origin making an angle θ with the positive x −axis, $\tan\theta = m$ so $\theta = \tan^{-1} m$. Therefore, $\tan^{-1} k - \tan^{-1} 2 = \tan^{-1} 2 - \tan^{-1} 1$. This means $\tan^{-1} k + \tan^{-1} 1 = 2\tan^{-1} 2$.

Take the tangent of both sides and apply the tangent sum and tangent double angle formulas:

$$\frac{\tan(\tan^{-1} k) + \tan(\tan^{-1} 1)}{1 - \tan(\tan^{-1} k)\tan(\tan^{-1} 1)} = \frac{2\tan(\tan^{-1} 2)}{1 - (\tan(\tan^{-1} 2))^2}$$

This simplifies to $\frac{k+1}{1-k} = \frac{4}{1-2^2}$. Cross-multiply: $-3(k+1) = 4(1-k)$. Simplify to $-3k - 3 = 4 - 4k$ or $k = 7$.