Directions: Unless otherwise stated, the domains and ranges of all functions are the **complex numbers**. Use the principal branch for the logarithms of complex numbers and for inverse trigonometric functions.

We adopt i as the imaginary unit: $i^2 = -1$. For a complex number $z = a + bi$ with real a and b, the notation $\mathfrak{Re}(z) = a$, $\mathfrak{Im}(z) = b$, $|z| = \sqrt{a^2 + b^2}$, $\arg(z) \in (-\pi, \pi]$ will denote the principal argument of z, and $\bar{z} = a - bi$: the complex conjugate of z. For a complex-valued matrix A, the notation A^* will denote the conjugate transpose of A such that $a^*_{kl} = \bar{a}_{lk}$ (apply the complex conjugate to all elements of A^T). For an $n \times n$ matrix M, $\lambda(M)$ is taken to mean the set of eigenvalues of M with (unordered) elements $\lambda_1, \lambda_2, \ldots, \lambda_n$. The symbols $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ represent the integers, real numbers, and complex numbers, respectively. By $\sqrt[n]{z}$, we refer to the principal n-th root of z (e.g., $\sqrt[5]{-32} = 2e^{\pi i/5}$). The following identity will be useful throughout this test: $e^{ix} = \cos(x) + i\sin(x)$.

Select **(E)** NOTA if none of the other answer choices for the question are correct.

1. The quadratic polynomial $p(z)$ has real coefficients. Suppose $p(1) = 10$ and $p(2 - i) = 0$. Find $p(2)$.

(A) 1 **(B)** 5 **(C)** 10 **(D)** 17 **(E)** NOTA

2. Let $A = \begin{bmatrix} 1-i & 2i \\ 3-2i & 2+i \end{bmatrix}$. Compute $\det(A)$.

(A) $-1-7i$ **(B)** $-1+5i$ **(C)** $7-7i$ **(D)** $7+5i$ **(E)** NOTA

3. Using the matrix A from the previous question, compute $\det(A^*)$.

(A) $-1-5i$ **(B)** $-1+7i$ **(C)** $7-5i$ **(D)** $7+7i$ **(E)** NOTA

4. If $|z + \bar{z}| = 6$ and $|z - \bar{z}| = 8$, then what is $|z^2|$?

(A) 100 **(B)** 64 **(C)** 36 **(D)** 25 **(E)** NOTA

5. For a square matrix A, a non-zero vector $\vec{v}$ is an eigenvector of A with associated eigenvalue λ if $A\vec{v} = \lambda\vec{v}$. Eigenvalues then, are the solutions to the equation $\det(\lambda I - A) = 0$, where I is the appropriately sized identity matrix. Let $A = \begin{bmatrix} 1-2i & 4 \\ 1-i & 3+2i \end{bmatrix}$. If $\lambda(A) = \{\lambda_1, \lambda_2\}$ and $|\lambda_1| > |\lambda_2|$, find $|2\lambda_1 - \lambda_2|$.

(A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** NOTA

6. Using the matrix A from the previous question, suppose $\vec{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ is an eigenvector associated to λ_1 (as defined in the previous question). Calculate $\dfrac{b}{a}$.

(A) $\dfrac{1+i}{2}$ **(B)** $\dfrac{1-i}{2}$ **(C)** $1+i$ **(D)** $1-i$ **(E)** NOTA

7. Suppose z is such that $\arg(z + 2 - 2i) = \dfrac{\pi}{12}$. Find the smallest possible value of $|z|^2$.

(A) 6 **(B)** 8 **(C)** 24 **(D)** 32 **(E)** NOTA

8. For complex arguments, the sine and cosine functions can take values outside the normal range of $[-1, 1]$. Which of the following is equal to $\sin(z)$?

(A) $\dfrac{e^{iz}+e^{-iz}}{2}$ **(B)** $\dfrac{e^{iz}-e^{-iz}}{2}$ **(C)** $\dfrac{e^{iz}+e^{-iz}}{2i}$ **(D)** $\dfrac{e^{iz}-e^{-iz}}{2i}$ **(E)** NOTA

9. For z to satisfy $\cos(z) = 2$, which of the following must be true? Note: $k \in \mathbb{Z}$

(A) $\mathfrak{Re}(z) = 0$
(B) $\mathfrak{Re}(z) = (2k+1)\pi$
(C) $|\mathfrak{Im}(z)| = \ln(2+\sqrt{3})$
(D) $|\mathfrak{Im}(z)| = 2k\pi + \ln(2+\sqrt{3})$
(E) NOTA

10. A matrix A is Hermitian if $A^* = A$. Suppose A is an $n \times n$ Hermitian matrix. Which of the following is not true?

(A) A has real diagonal entries.
(B) $AA^* = A^*A$
(C) A has only real eigenvalues.
(D) $\det(A) = \det(A^*)$
(E) NOTA

11. The locus of points z in the complex plane which satisfy $|z+6i| + |z-8| = 26$ forms a closed curve $\mathcal{C}$. Find the area of the region enclosed by $\mathcal{C}$.

(A) 156π **(B)** 288π **(C)** 312π **(D)** 360π **(E)** NOTA

12. The inner product is the extension of the dot product to complex-valued vectors. For two vectors $\vec{u}, \vec{v} \in \mathbb{C}^n$, the inner product is defined as $\langle \vec{u}, \vec{v} \rangle = \sum_{k=1}^{n} u_k \bar{v}_k = \vec{v}^* \vec{u}$. The vectors $\vec{u}$ and $\vec{v}$ are orthogonal, then, if $\langle \vec{u}, \vec{v} \rangle = 0$.

Suppose $\vec{u} = \begin{bmatrix} 1 \\ z \\ 2-i \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2-4i \\ 1+3i \\ z \end{bmatrix}$, for some complex number z. If $\vec{u}$ and $\vec{v}$ are orthogonal, find $|z|^2$.

(A) 13 **(B)** 17 **(C)** 20 **(D)** 25 **(E)** NOTA

13. A square matrix A is unitary if $AA^* = I$, where I is the appropriately sized identity matrix. Suppose A is an $n \times n$ unitary matrix. Which of the following is not necessarily true?

(A) There exists a positive integer k such that $A^k = I$, where I is the $n \times n$ identity matrix.
(B) If $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of A corresponding to different eigenvalues, then $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.
(C) The magnitude of the trace of A is at most n.
(D) For all positive integers k, A^k is unitary.
(E) NOTA

14. Which of the following is equal to $\arg(-8+15i)$?

(A) $\arctan\left(-\dfrac{8}{15}\right)$
(B) $-\pi - \arcsin\left(\dfrac{15}{17}\right)$
(C) $\dfrac{\pi}{2} + \arccos\left(\dfrac{8}{17}\right)$
(D) $\pi + \operatorname{arccot}\left(-\dfrac{15}{8}\right)$
(E) NOTA

15. For complex $z = x + yi$ with real x and y, the value z can be represented by the matrix $Z = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$. Which of the following statements is not true?

(A) If $|z| = 1$, then Z is a rotation matrix.

(B) $\lambda(Z) = \{z, \bar{z}\}$

(C) $\det(Z) = |z|$

(D) $\begin{bmatrix} 1 \\ i \end{bmatrix}$ is an eigenvector for all such matrices Z.

(E) NOTA

16. Let S be the set of complex numbers w for which $\tan(z) = w$ has no solution. If s_1 and s_2 are elements of S, then find the largest possible value of $|s_1 - s_2|$.

(A) 0
(B) 1
(C) 2
(D) $|s_1 - s_2|$ is unbounded
(E) NOTA

17. A 2×2 matrix contains the numbers 1, i, -1, and $-i$, in some order. Find the probability that the matrix is invertible.

(A) 0 **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{2}$ **(E)** NOTA

18. Find the number of distinct values of $0 \leq z < 2\pi$ such that z is a solution to at least one of $z^{20} = 1$ or $z^{25} = 1$.

(A) 40 **(B)** 45 **(C)** 100 **(D)** 500 **(E)** NOTA

19. Using the fact that $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, evaluate $\det\left(\sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}^n\right)$. Hint: Use question 15.

(A) 1 **(B)** e **(C)** 3 **(D)** $3e$ **(E)** NOTA

20. If $\sum_{n=0}^{2025} ni^n = a + bi$ for real a and b, find $a + b$.

(A) -2025 **(B)** -1 **(C)** 1 **(D)** 2025 **(E)** NOTA

21. Ramanujan's constant, $\mathcal{R} = e^{\pi\sqrt{163}}$, is approximately equal to 262537412640768743.99999999999925. Determine which quadrant of the complex plane the complex number $e^{\mathcal{R}i\pi}$ lies in.

(A) I **(B)** II **(C)** III **(D)** IV **(E)** NOTA

Use the following information to answer questions 22 through 24:

The limit of a complex-valued function f at point c is defined as the unique value L, if it exists, such that for all real $\epsilon > 0$ there exists real $\delta > 0$ such that $|f(z) - L| < \epsilon$ for all $z : 0 < |z - c| < \delta$. This relation is denoted as $L = \lim_{z \to c} f(z)$. We say f is continuous at c if $f(c) = \lim_{z \to c} f(z)$, provided both values exist.

22. Let $f(z) = \dfrac{z^2+1}{z-i}$. Find $\lim_{z \to i} f(z)$.

(A) 0 **(B)** i **(C)** $2i$ **(D)** Does not exist **(E)** NOTA

23. Let $f(z) = \dfrac{z^2+1}{z-i}$. If $L = \lim_{z \to i} f(z)$ and $\epsilon = \dfrac{1}{10}$, find the largest value of δ for which $|f(z) - L| < \epsilon$ for all $|z - i| < \delta$.

(A) 0.01 **(B)** 0.05 **(C)** 0.1 **(D)** 0.2 **(E)** NOTA

24. A function $f(z)$ is continuous over a set S if f is continuous at c for all $c \in S$. Which of the following functions is continuous over $\mathbb{R}$, but not over $\mathbb{C}$? Remember, the domains and ranges of all functions are $\mathbb{C}$, excluding where f is undefined. This means you should take all limits using the provided definition, even if $c \in \mathbb{R}$.

(A) $f(z) = \sqrt[3]{z}$

(B) $f(z) = \tan(z)$

(C) $f(z) = \sin^2(z)$

(D) $f(z) = \dfrac{1}{2+\sin(z)}$

(E) NOTA

25. Let $A = \begin{bmatrix} 1-2i & 4 \\ 1-i & 3+2i \end{bmatrix}$. There exists a unique complex number w and finite, non-zero, complex-valued vector $\vec{u}$ such that $\lim_{n \to \infty} \dfrac{A^n}{w^n} \begin{bmatrix} 1+i \\ 1-i \end{bmatrix} = \vec{u}$. For the limit of a sequence of vectors, take the limit of the sequences of components.

If $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, compute $|a+b+w|^2$.

(A) 2 **(B)** 5 **(C)** 8 **(D)** 10 **(E)** NOTA

26. Determine which quadrant of the complex plane the complex number e^{i^i} lies in.

(A) I **(B)** II **(C)** III **(D)** IV **(E)** NOTA

27. The locus of points z in the complex plane for which $|z - 2025 - 2025i| = \dfrac{|\mathfrak{Re}(z)|}{2025} \times |2025 - \tan(\arg(z))|$ is in what shape?

(A) Ellipse **(B)** Hyperbola **(C)** Parabola **(D)** Two lines **(E)** NOTA

28. The largest possible value of $|1+z|^2$, for $z \in \mathbb{C}$ such that $|1+z^2| \le \sqrt{3}$, can be written as $m + \sqrt{n}$, for positive integers m and n. Find $m+n$.

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** NOTA

29. Which of the following is equivalent to $\sum_{n=0}^{90}\sum_{m=0}^{n}\sin((m+n)^\circ)$? Hint: this is a complex numbers test.

(A) $\dfrac{2\cot(1^\circ)+2\cot(0.5^\circ)+\cot^2(0.5^\circ)}{4}$

(B) $\dfrac{2\csc(1^\circ)+2\cot(0.5^\circ)+\cot^2(0.5^\circ)}{4}$

(C) $\dfrac{2\cot(1^\circ)+2\cot(0.5^\circ)+\csc^2(0.5^\circ)}{4}$

(D) $\dfrac{2\csc(1^\circ)+2\cot(0.5^\circ)+\csc^2(0.5^\circ)}{4}$

(E) NOTA

30. Evaluate $\sum_{k=1}^{\infty}\mathfrak{Re}\left(\ln\left(1+\frac{i^{2k+1}}{k}\right)\right)$. You may find the following information useful:

For positive integers n, $\Gamma(n)=(n-1)!$. For complex numbers $z\notin\mathbb{Z}_{\leq 1}$, $\Gamma(z)=\dfrac{1}{z}\prod_{n=1}^{\infty}\dfrac{\left(1+\frac{1}{n}\right)^z}{1+\frac{z}{n}}=(z-1)\Gamma(z-1)$, and if z is not an integer, $\Gamma(z)\Gamma(1-z)=\dfrac{\pi}{\sin(\pi z)}$.

(A) $\dfrac{\ln(e^{2\pi}-1)-\ln(2\pi)-\pi}{2}$

(B) $\dfrac{\ln(e^{2\pi}+1)-\ln(2\pi)-\pi}{2}$

(C) $\ln(e^{2\pi}-1)-\ln(\pi)$

(D) $\ln(e^{2\pi}+1)-\ln(\pi)$

(E) NOTA