Answers:

1. B
2. A
3. B
4. D
5. C

6. A
7. A
8. D
9. C
10. E

11. A
12. C
13. A
14. E
15. C

16. C
17. C
18. A
19. E
20. D

21. D
22. C
23. C
24. D
25. D

26. E
27. B
28. D
29. C
30. A

Solutions:

1. $\boxed{\text{B}}$: $p(z) = k(z-(2-i))(z-(2-i)) = k(z^2-4z+5)$. $p(1) = 2k = 10 \implies k = 5 \implies p(2) = 5(4-8+5) = 5$.

2. $\boxed{\text{A}}$: $\det(A) = (1-i)(2+i) - 2i(3-2i) = 3-i-(4+6i) = -1-7i$.

3. $\boxed{\text{B}}$: $\det(A^*) = (1+i)(2-i) - (-2i)(3+2i) = 3+i-(4-6i) = -1+7i$.
In general, if $D = \det(A)$, then $\det(A^*) = \bar{D}$.

4. $\boxed{\text{D}}$: $z+\bar{z} = 2\mathfrak{Re}(z)$ and $z-\bar{z} = 2\mathfrak{Im}(z)$, so $|\mathfrak{Re}(z)| = 3$ and $|\mathfrak{Im}(z)| = 4$. Hence, $|z^2| = |z|^2 = 25$.

5. $\boxed{\text{C}}$: $\det(\lambda I - A) = (\lambda-(1-2i))(\lambda-(3+2i)) - 4(1-i) = \lambda^2 - 4\lambda + 7 - 4i - (4-4i) = (\lambda-3)(\lambda-1)$.
The roots are $\lambda_1 = 3$ and $\lambda_2 = 1$, so $|2\lambda_1 - \lambda_2| = 5$.

6. $\boxed{\text{A}}$: We'll use Gaussian elimination to solve $3I - A = \vec{0}$. We have

$$\left[\begin{array}{cc|c} 2+2i & -4 & 0 \\ -1+i & -2i & 0 \end{array}\right] \xrightarrow{R_1+2iR_2 \to R_1} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ -1+i & -2i & 0 \end{array}\right].$$

In particular, $a(-1+i) - 2ib = 0 \implies \dfrac{b}{a} = \dfrac{-1+i}{2i} = \dfrac{1+i}{2}$.

7. $\boxed{\text{A}}$: Convert to Cartesian coordinates. Let $A = (0,0)$, $B = (2,-2)$, and C be a point on the line $y = \tan\left(\dfrac{\pi}{12}\right)x$ in the first quadrant. The vector $\overrightarrow{BC}$ corresponds to z and $\overrightarrow{AB}$ to $2-2i$. Notice $\angle CAB = \dfrac{\pi}{3}$ and $AB = 2\sqrt{2}$. Let $\theta = \angle BCA$. By the law of sines, $\dfrac{BC}{\sin(\pi/3)} = \dfrac{2\sqrt{2}}{\sin(\theta)} \implies BC = \sqrt{6}\csc(\theta)$, and since $\csc(\theta) \geq 1$, $|z^2| = BC^2 \geq 6$.

8. $\boxed{\text{D}}$: Using the identity $e^{iz} = \cos(z) + i\sin(z)$ and $e^{-iz} = \cos(z) - i\sin(z) \implies \dfrac{e^{iz}-e^{-iz}}{2i} = \dfrac{2i\sin(z)}{2i} = \sin(z)$.

9. $\boxed{\text{C}}$: We have $\cos(z) = \dfrac{e^{iz}+e^{-iz}}{2} = 2 \implies e^{2iz} - 4e^{iz} + 1 = 0$. Then,

$$e^{2iz} - 4e^{iz} + 4 = 3 \implies e^{iz} = 2 \pm \sqrt{3} \implies z = 2\pi k + \ln(2\pm\sqrt{3})i;\ k \in \mathbb{Z}$$

Since $\dfrac{1}{2-\sqrt{3}} = 2+\sqrt{3}$, we have $|\mathfrak{Im}(z)| = \ln(2+\sqrt{3})$.

10. $\boxed{\text{E}}$: A is true: $a^*_{mm} = \bar{a}_{mm} = a_{mm} \implies a_{mm} \in \mathbb{R}$.
B is true: $AA^* = AA = A^*A$.
C is true: For $\vec{v} \in \mathbb{C}^n$, notice $(\vec{v}^*A\vec{v})^* = \vec{v}^*A^*\vec{v} = \vec{v}^*A\vec{v} \implies \vec{v}^*A\vec{v} \in \mathbb{R}$. Suppose $\vec{v}$ is an eigenvector of A with associated eigenvalue λ. Then, $\vec{v}^*A\vec{v} = \lambda\vec{v}^*\vec{v} = \lambda||\vec{v}||^2 \in \mathbb{R} \implies \lambda \in \mathbb{R}$.
D is trivially true.

11. $\boxed{\text{A}}$: The sum of the distances from z to the points $-6i$ and 8 are a constant: this describes an ellipse. The distance between the foci is 10, so $c = 5$. At an endpoint of the minor axis, the distance from that point to each focus is the same and is the length of the major radius, 13. Hence, $b = \sqrt{a^2-c^2} = 12$ and the area of the ellipse is $\pi ab = 156\pi$.

12. $\boxed{\text{C}}$: Set $z = x + yi$, for real x and y. Then,

$$\langle \vec{u}, \vec{v} \rangle = 2 + 4i + (x + yi)(1 - 3i) + (2 - i)(x - yi) = (3x + 2y + 2) + (-4x - y + 4)i = 0$$

So, $\begin{cases} 3x + 2y = -2 \\ -4x - y = -4 \end{cases} \implies -5x = -10 \implies x = 2, y = -4$, so $z = 2 - 4i \implies |z|^2 = 20.$

13. $\boxed{\text{A}}$: Notice immediately that $A^*A = (AA^*)^* = I^* = I$.

Let $\vec{v}$ be an eigenvector of A with associated eigenvalue λ. Since $A\vec{v} = \lambda\vec{v}$, we also have $\vec{v}^*A^* = \bar{\lambda}\vec{v}^*$, so $\vec{v}^*A^*A\vec{v} = \lambda\bar{\lambda}v^*\vec{v} \implies ||\vec{v}||^2 = |\lambda|^2||\vec{v}||^2 \implies |\lambda| = 1$. Statement C immediately follows by the triangle inequality. Suppose $\vec{u}$ is an eigenvector of A with associated eigenvalue $\mu \neq \lambda$. Then $\vec{u}^*A^*A\vec{v} = \langle \vec{v}, \vec{u} \rangle = \bar{\mu}\lambda\langle \vec{v}, \vec{u} \rangle$. Hence, either $\bar{\mu}\lambda = 1$, which is impossible as $|\mu| = |\lambda| = 1$ and $\mu \neq \lambda$, or $\langle \vec{v}, \vec{u} \rangle = 0$, so $\vec{u}$ and $\vec{v}$ are orthogonal, and statement B is true.

Since all eigenvalues are of magnitude 1, we know $|\det(A)| = 1 \implies \det(A) = e^{q\pi i}$. However, if $q \notin \mathbb{Q}$, then $\det(A^k) = \det(A)^k = e^{q\pi k i} \neq 1 = e^{2\pi m i}; m \in \mathbb{Z}$, as $qk \notin \mathbb{Z}$ due to q being irrational. Statement A is therefore false.

Statement D follows by induction from the fact that if B is also unitary and $n \times n$ then AB is unitary. To see this, note that $AB(AB)^* = ABB^*A^* = AIA^* = AA^* = I$. In general, $n \times n$ unitary matrices form a group.

14. $\boxed{\text{E}}$: The target angle is in quadrant II. We have $\arctan(\theta) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\arcsin(\theta) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so for a quadrant II angle, the inverse tangent alone cannot reach, ruling out A. For choice B, the angle is coterminal, but is less than $-\pi$, so it cannot be the principal argument. For choice C, the resulting angle has sine equal to $\frac{8}{17}$, which does not match our target angle. Finally, since $\text{arccot}(\theta) \in (0, \pi)$, we have $\arg(-8 + 15i) = \text{arccot}\left(-\frac{15}{8}\right)$, but since choice D adds π, the resulting angle ends up in the fourth quadrant.

None of the given angles match.

Remark: The preamble to the test specifies $\arg(z) \in (-\pi, \pi]$. Do not accept disputes arguing B is correct as $-\pi - \arcsin\left(\frac{15}{17}\right) < -\pi$.

15. $\boxed{\text{C}}$: A: True. Note $x = \cos(\theta), y = \sin(\theta)$ makes Z a rotation matrix.

B: True: $\det(A - \lambda I) = (x - \lambda)^2 + y^2 = 0 \implies \lambda = x \pm yi = \{z, \bar{z}\}$. This set contains one element if $z = \bar{z}$ and that element is a repeat eigenvalue.

C: False: $\det(Z) = x^2 + y^2 = |z|^2$

D: True: $\begin{bmatrix} x & -y \\ y & x \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} x - yi \\ y + xi \end{bmatrix} = (x - yi) \begin{bmatrix} 1 \\ i \end{bmatrix}$

16. $\boxed{\text{C}}$: The range of $\tan(z)$ is $\mathbb{C}\backslash\{i, -i\}$. Using the complex definitions of the sine and cosine, we have $\frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})} = w$.

Let $u = e^{iz}$. Then, $u^2 - 1 = wi(u^2 + 1) \implies u^2(wi - 1) = -1 - wi \implies u = e^{iz} = \pm\sqrt{\frac{w - i}{w + i}}i$.

Clearly, if $w = -i$, we have a division by zero. If $w = i$, then $e^{iz} = 0$, which is also impossible.

Otherwise, $\sqrt{\frac{w - i}{w + i}}i$ defines a complex number with nonzero magnitude, so its logarithm can be taken, providing a solution. It follows that $|s_1 - s_2| = 2$.

17. $\boxed{\text{C}}$: Note that the entries of the matrix are i^0, i^1, i^2, and i^3. If the product of the entries on the main diagonal is i^n, then the product of the entries on the anti-diagonal is i^{6-n}. These exponent pairs will be either 1 and 5, 2 and 4, or 3 and 3. Only in the middle $\frac{1}{3}$ of these cases is the determinant not 0.

18. $\boxed{\text{A}}$: Trivially, the first equation has 20 solutions and the second equation has 25 solutions. However, values of z where $z^5 = 1$ are double-counted, for an actual total of 40 distinct solutions.

19. $\boxed{\text{E}}$: The matrix $\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$ corresponds to the complex number $1-3i$, so $\sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}^n$ evaluates to the complex number corresponding to $e^{1-3i} = e(\cos(3) - i\sin(3))$, that is, $e\begin{bmatrix} \cos(3) & \sin(3) \\ -\sin(3) & \cos(3) \end{bmatrix}$ which has determinant $e^2(\cos^2(3) + \sin^2(3)) = e^2$.

20. $\boxed{\text{D}}$: The sum of the coefficients of any four consecutive terms (starting with k being 1 more than a multiple of 4) is $k - (k+1) - (k+2) + (k+3) = 0$. Only the $n = 2025$ term is not part of such a group, so the sum of the coefficients of the sum is 2025.

21. $\boxed{\text{D}}$: $\mathcal{R}$ is an interesting number, but the only thing that matters here is that it is very slightly less than a multiple of 2, so the argument of the desired number is very slightly less than 2π, putting it in Quadrant IV.

22. $\boxed{\text{C}}$: For $z \neq i$, $f(z) = z + i$, and L is obviously $2i$.

23. $\boxed{\text{C}}$: For $z \neq i$, $|f(z) - L| = |z + i - 2i| = |z - i| < \frac{1}{10} = \epsilon$, by the definition of the limit. However, that is precisely in the form $|z - i| < \delta$, so the largest value of δ is $\frac{1}{10}$.

24. $\boxed{\text{D}}$: For choice A, if $z = re^{i\theta}$ for $r \in \mathbb{R}_{\geq 0}$, then $f(z) = \sqrt[3]{r}e^{i\theta/3}$, and importantly, $\theta \in (-\pi, \pi]$.

Consider $\lim_{z \to -1} f(z)$ and the path on $r = 1$ approaching -1. If $\mathfrak{Im}(z) > 0$, then $0 < \arg(f(z)) < \frac{\pi}{3}$, and as $z \to 0$, $f(z) \to e^{\pi i/3}$, but if $\mathfrak{Im}(z) < 0$, then $-\frac{\pi}{3} < \arg(f(z)) < 0$, and as $z \to 0$, $f(z) \to e^{-\pi i/3}$. Hence, $\lim_{z \to -1} f(z)$ is undefined and $\sqrt[3]{z}$ is not continuous on $\mathbb{R}$.

B is obviously not continuous on $\mathbb{R}$ as $\tan(\pi/2)$ is undefined.

C is obviously continuous on $\mathbb{C}$.

For choice D, using the complex definition of the sine, $f(z) = \frac{2i}{2i + e^{iz} - e^{-iz}}$, and if $z \in \mathbb{R}$, this is obviously continuous. However, there does exist a complex value $z : \sin(z) = -2$, so f is not continuous on $\mathbb{C}$.

25. $\boxed{\text{D}}$: Recall from questions 5 and 6 that $\lambda(A) = \{1, 3\}$ and that $\vec{v}_1 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$ is an eigenvector associated to $\lambda = 3$. Using an analogous row reduction procedure with $\lambda_2 = 1$, we have

$$\left[\begin{array}{cc|c} 2i & -4 & 0 \\ -1+i & -2-2i & 0 \end{array}\right] \xrightarrow{R_2 - \frac{1}{2}(1+i)R_1 \to R_2} \left[\begin{array}{cc|c} 2i & -4 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Hence, $\vec{v}_2 = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$ is an eigenvector associated with λ_2. Notice $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

Hence, $\begin{bmatrix} 1+i \\ 1-i \end{bmatrix} = \vec{u}_1 + \vec{u}_2$, for eigenvectors $\vec{u}_1$ and $\vec{u}_2$, corresponding to eigenvalues of 3 and 1, respectively. Thus, $A^n \begin{bmatrix} 1+i \\ 1-i \end{bmatrix} = A^n\vec{u}_1 + A^n\vec{u}_2 = 3^n\vec{u}_1 + \vec{u}_2 \implies \lim_{n\to\infty} \frac{A^n}{w^n} \begin{bmatrix} 1+i \\ 1-i \end{bmatrix} = \lim_{n\to\infty} \frac{3^n\vec{u}_1 + \vec{u}_2}{w^n}$, which is finite and non-zero, evaluating to $\vec{u}_1$ only if $w = 3$.

We now only need $\vec{u}_1$. Set $x\vec{v}_1 = \vec{u}_1$ and $y\vec{v}_2 = \vec{u}_2$. Consider the equation $x\vec{v}_1 + y\vec{v}_2 = \begin{bmatrix} 1+i \\ 1-i \end{bmatrix}$. Writing this system in augmented matrix form (left column is $\vec{v}_2$) gives:

$$\left[\begin{array}{cc|c} 2i & 1-i & 1+i \\ 1 & 1 & 1-i \end{array}\right] \xrightarrow{R_2 + \frac{i}{2}R_1 \to R_2} \left[\begin{array}{cc|c} 2i & 1-i & 1+i \\ 0 & \frac{3+i}{2} & \frac{1-i}{2} \end{array}\right].$$

So, $x = \dfrac{1-i}{3+i} = \dfrac{1-2i}{5} \implies \vec{u} = \dfrac{1}{5}\begin{bmatrix} -1-3i \\ 1-2i \end{bmatrix} \implies |a+b+w|^2 = |3-i|^2 = 10.$

Remark: This process of iteratively left-multiplying a vector by a matrix A and normalizing to iteratively approach an eigenvector of A corresponding to the largest magnitude eigenvalue is known as the power method. It is useful for numerically approximating the largest eigenvalue of a high-dimensional matrix where analytically calculating the eigenvalues is infeasible. This question shows a method to find other eigenvalues by translating the matrix A.

26. $\boxed{\textbf{E}}$: Consider the exponent, i^i. Noting that $e^{i\pi/2} = i$, this is $(e^{i\pi/2})^i = e^{i^2\pi/2} = e^{-\pi/2}$, a real number. Similarly, $e^{e^{-\pi/2}}$ is thus a real number and lies on the real axis, not in any quadrant.

27. $\boxed{\textbf{B}}$: Let $p = 2025(1+i)$ and $m = 2025$. We have $|z-p| = \dfrac{|\mathfrak{Re}(z) - m\mathfrak{Im}(z)|}{m}$.

Immediately observe that the LHS is the distance from z to p. Set $z = x+iy$. The RHS becomes $\dfrac{|x-my|}{m} = \dfrac{|x-my|\sqrt{m^2+1}}{m\sqrt{m^2+1}}$. Notice that $\dfrac{|x-my|}{m} = \dfrac{|x-my|}{\sqrt{m^2+1}}$ is the distance from (x,y) to the line $y = -mx$. Hence, the locus of points z satisfying $\dfrac{|z-p|\sqrt{m^2+1}}{|\mathfrak{Re}(z) - m\mathfrak{Im}(z)|} = \dfrac{\sqrt{m^2+1}}{m} = \epsilon$ is the locus of points z for which the ratio of the distance from z to p to the distance from z to a line is the ratio ϵ. This is the definition of a conic section with eccentricity ϵ, and since $\epsilon = \dfrac{\sqrt{m^2+1}}{m} > 1$, we have a hyperbola.

28. $\boxed{\textbf{D}}$: Let $z = x+yi$ for $x, y \in \mathbb{R}$.

We have $|1+z^2|^2 = |x^2-y^2+1+2xyi|^2 = x^4 - 2x^2y^2 + y^4 + 2(x^2-y^2) + 1 + 4x^2y^2$
$= (x^2+y^2)^2 - 2(x^2+y^2) + 1 + 4x^2 = (x^2+y^2-1)^2 + 4x^2 \le 3.$

By QM-AM, $\dfrac{x^2+y^2-1+2x}{2} \le \sqrt{\dfrac{(x^2+y^2-1)^2+(2x)^2}{2}} \le \dfrac{\sqrt{6}}{2} \implies x^2+2x+y^2 \le 1+\sqrt{6}$

$\implies |1+z|^2 = (1+x)^2 + y^2 = x^2+2x+y^2+1 \le 2+\sqrt{6} \implies m+n = 2+6 = 8.$

For QM-AM to hold in equality, we need $x^2+y^2-1 = 2x \iff (x-1)^2+y^2 = 2$, which does intersect with the boundary $(x^2+y^2-1)^2 + 4x^2 = 3$ at two points, thus this maximum is attainable.

In particular, $z = \dfrac{\sqrt{6}}{4} + \dfrac{\sqrt{10+8\sqrt{6}}}{4}i$ works.

Remark: QM-AM asserts that for positive real numbers a and b, $\dfrac{a+b}{2} \le \sqrt{\dfrac{a^2+b^2}{2}}$ with equality iff $a = b$. To see this, note $\left(\dfrac{a+b}{2}\right)^2 \le \dfrac{a^2+b^2}{2} \iff \dfrac{a^2-2ab+b^2}{4} = \dfrac{(a-b)^2}{4} \ge 0$, which is obviously true.

29. $\boxed{\text{C}}$: There are two series that will need to be evaluated beforehand. In what follows, we omit the degree symbol for the trigonometric arguments.

First, $\displaystyle\sum_{n=0}^{90}\sin(n) = \frac{1}{2i}\sum_{n=0}^{90}(e^{in}-e^{-in}) = \frac{1}{2i}\left(\frac{e^{91i}-1}{e^{i}-1}-\frac{e^{-91i}-1}{e^{-i}-1}\right)$

$$= \frac{1}{2i}\left(\frac{(e^{91i}-1)(e^{-i}-1)-(e^{-91i}-1)(e^{i}-1)}{2-e^{i}-e^{-i}}\right) = \frac{1}{2i}\left(\frac{(1+i-e^{91i}-e^{-i})-(1-i-e^{i}-e^{-91i})}{2-e^{i}-e^{-i}}\right)$$

$$= \frac{1}{2i}\left(\frac{2i+e^{i}-e^{-i}+e^{-91i}-e^{91i}}{2-e^{i}-e^{-i}}\right) = \frac{1+\sin(1)-\sin(91)}{2(1-\cos(1))} = \frac{1+\sin(1)-\cos(1)}{2(1-\cos(1))} = \frac{1}{2}+\frac{\sin(1)}{2(1-\cos(1))} = \frac{1+\cot(1/2)}{2}.$$

Similarly, $\displaystyle\sum_{n=0}^{90}\sin(2n) = 1+2\sum_{i=0}^{44}\sin(2n) = 1+\frac{1}{i}\left(\frac{e^{90i}-1}{e^{2i}-1}-\frac{e^{-90i}-1}{e^{-2i}-1}\right)$

$= 1+\dfrac{1}{i}\left(\dfrac{-1+i}{e^{2i}-1}-\dfrac{-1-i}{e^{-2i}-1}\right)$. If $z=\dfrac{-1+i}{-1+e^{2i}}$, then we have $1+\dfrac{1}{i}(z-\bar{z}) = 1+2\mathfrak{Im}(z)$

$$= 1+2\mathfrak{Im}\left(\frac{(-1+i)(-1+e^{-2i})}{(-1+e^{2i})(-1-e^{-2i})}\right) = 1+2\mathfrak{Im}\left(\frac{1-i-e^{-2i}+ie^{-2i}}{2-e^{2i}-e^{-2i}}\right)$$

$$= 1+2\mathfrak{Im}\left(\frac{-i-i\sin(-2)+i\cos(-2)}{2(1-\cos(2))}\right) = 1+\frac{-1+\sin(2)+\cos(2)}{1-\cos(2)} = \frac{\sin(2)}{1-\cos(2)} = \cot(1).$$

Now consider the main problem. We have $\displaystyle\sum_{n=0}^{90}\sum_{m=0}^{90}\sin(m+n) = 2\sum_{n=0}^{90}\sum_{m=0}^{n}\sin(m+n)-\sum_{n=0}^{90}\sin(2n)$

and $\displaystyle\sum_{n=0}^{90}\sum_{m=0}^{90}\sin(m+n) = \sum_{n=0}^{90}\sum_{m=0}^{90}(\sin(m)\cos(n)+\cos(m)\sin(n) = 2\left(\sum_{n=0}^{90}\sin(n)\right)^2$.

Then we have $\displaystyle\sum_{n=0}^{90}\sum_{m=0}^{n}\sin(m+n) = \left(\sum_{n=0}^{90}\sin(n)\right)^2+\frac{1}{2}\sum_{n=0}^{90}\sin(2n)$

$$= \frac{1+2\cot(1/2)+\cot^2(1/2)}{4}+\frac{\cot(1)}{2} = \frac{2\cot(1)+2\cot(1/2)+\csc^2(1/2)}{4}.$$

30. $\boxed{\text{A}}$: Let S be the value of the series. We have $S=\displaystyle\sum_{k=1}^{\infty}\ln\left|1+i\frac{(-1)^k}{k}\right|$

$= \displaystyle\sum_{k=1}^{\infty}\ln\left(\sqrt{1+\frac{1}{k^2}}\right) = \frac{1}{2}\ln\left(\prod_{k=1}^{\infty}\left(1+\frac{1}{k^2}\right)\right)$. Let $P=\displaystyle\prod_{k=1}^{\infty}\left(1+\frac{1}{k^2}\right)$.

Then, $\displaystyle\frac{1}{P} = \prod_{k=1}^{\infty}\frac{1}{1+\frac{1}{k^2}} = \prod_{k=1}^{\infty}\frac{1}{\left(1+\frac{i}{k}\right)\left(1-\frac{i}{k}\right)} = \prod_{k=1}^{\infty}\frac{\left(1+\frac{1}{k}\right)^{i}\left(1+\frac{1}{k}\right)^{-i}}{\left(1+\frac{i}{k}\right)\left(1-\frac{i}{k}\right)}$, after multiplying through by a constant 1.

Separating the products gives $\displaystyle\frac{1}{P} = \prod_{k=1}^{\infty}\frac{\left(1+\frac{1}{k}\right)^{i}}{\left(1+\frac{i}{k}\right)}\prod_{k=1}^{\infty}\frac{\left(1+\frac{1}{k}\right)^{-i}}{\left(1-\frac{i}{k}\right)}$

$= i\Gamma(i)(-i)\Gamma(-i) = \Gamma(i)\Gamma(-i) = \dfrac{1}{i}\Gamma(1+i)\Gamma(-i)$, by the recursive definition of the Gamma function.

Then, $\dfrac{1}{P} = \dfrac{\pi}{i\sin(-\pi i)} = \dfrac{\pi}{\sinh(\pi)} \implies P = \dfrac{\sinh(\pi)}{\pi}$

$$\implies S = \frac{1}{2}\ln\left(\frac{1}{\pi}\times\frac{e^{2\pi}-1}{2e^{\pi}}\right) = \frac{\ln(e^{2\pi}-1)-\ln(2\pi)-\pi}{2}.$$