Answers: BDCCD ECACC DCBCA BBCCD DDDAC CBCAE

Solutions

1. We see that $r = 4$ and $\theta = \frac{2\pi}{3}$. For rectangular form, we want $(r\cos\theta, r\sin\theta)$ which is $(4 \cdot -\frac{1}{2}, 4 \cdot \frac{\sqrt{3}}{2}) = \boxed{(-2, 2\sqrt{3})}$ **(B)**.

2. From double angle identity, $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$. One minus this quantity is $\boxed{2\sin^2\theta}$ **(D)**.

3. We see that

$$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\cos^2 x + \sin^2 x}{\cos x \sin x} = \frac{2}{\sin 2x}.$$

Then we want solutions to $\sin 2x = \frac{2}{3}$, of which there are $\boxed{4}$ in the interval $[0; 2\pi)$ by graphing.

4. Notice that $\sec\theta = \frac{\tan\theta}{\sin\theta} = \frac{1}{\cot\theta\sin\theta}$. Direct substitution yields $\frac{3}{2} \cdot -\frac{2}{\sqrt{5}} = \boxed{-\frac{3\sqrt{5}}{5}}$ **(C)**.

5. For this question, we need to make two observations. For extremely small x, both $\sin x$ and $\tan x$ are approximately x. However, $\sin x$ is always below x and $\tan x$ is above x. Hence, the correct answer will use a tangent (since $\sin x \approx \tan x \approx x$ reduces each answer choice to x). To decide which of these choices is correct, we notice that the relative error in $\tan x \approx x$, $\frac{\tan x - x}{x}$, increases going away from zero. Therefore, $\frac{1}{2}\tan 2x$ will have a greater error than $2\tan\frac{1}{2}x$, making it more above the line $y = x$. Hence the answer is $\boxed{\frac{1}{2}\tan 2x}$ **(D)**.

6. This question attempts to take the inverse sine of $\frac{13}{5} > 1$. Since sine is never greater than one, the operation is undefined and no answer exists. **(E)**.

7. This is a standard application of the Law of Sines. We have $\frac{BC}{\sin\angle A} = \frac{CA}{\sin\angle B}$. Solving for CA gives

$$CA = BC\frac{\sin\angle B}{\sin\angle A} = 18 \cdot \frac{\frac{\sqrt{2}}{2}}{\frac{1}{2}} = \boxed{18\sqrt{2}} \textbf{ (C)}.$$

8. To find the area of an n-sided inscribed regular polygon, we split it up into n isosceles triangles. Each of these triangles has an angle $\frac{2\pi}{n}$ enclosed by two sides of length 1. Then, the total area of the n triangular sections is $n \cdot \frac{1}{2}1 \cdot 1 \cdot \sin\frac{2\pi}{n}$. When $n = 5$, $\sin\frac{2\pi}{5} = \sqrt{1 - \cos^2\frac{2\pi}{5}} = \frac{\sqrt{10+2\sqrt{5}}}{4}$. Therefore the final result is $\boxed{\frac{5}{8}\sqrt{10+2\sqrt{5}}}$ **(A)**.

9. If 2θ is in the first quadrant, then either $\theta \in (0°; 45°)$ or $\theta \in (180°, 225°)$. If the first is the case, then 3θ cannot fall into the third quadrant (in the second, 3θ lies in the range $(180°, 315°)$). Therefore θ lies in quadrant $\boxed{\text{III}}$ **(C)**.

10. The expression is an infinite geometric series with first term $\sin x$ and common ratio $\sin x$. The overall sum is then equal to $\frac{\sin x}{1-\sin x}$. This is zero only whenever $\sin x = 0$. For the series to diverge, the common ratio must be ± 1, and so $\sin x = 1, -1$ are also considered. In total, these conditions occur a total of $\boxed{4}$ times in the interval $[0; 2\pi)$ **(C)**.

11. The expression $\sin x + \csc x$ is equal to $u + \frac{1}{u}$ where $u = \sin x$. Since $u \in [-1; 1]$, $u + \frac{1}{u} \in (-\infty; -2] \cup [2; \infty)$ and the answer is $\boxed{2}$. Alternatively, there is no way for $u + \frac{1}{u}$ to have absolute value below 2. In any case the answer is **(D)**.

12. This is just a knowledge question. If you don't remember the range of arcsec, we can use arccos instead since $\operatorname{arcsec} x = \arccos\frac{1}{x}$. This function has range $[0; \pi]$ and so the answer is $\boxed{\pi}$ **(C)**.

13. Lots of polar notation! We work from the inside out. $\exp\left(\frac{\pi}{4}i\right)$ is $\frac{\sqrt{2}}{2}(1+i)$. Multiplying by $\sqrt{2}$ leaves $1 + i$. We can split up $\exp\left(\frac{\pi}{4}(1+i)\right)$ into the real and imaginary parts, giving $e^{\frac{\pi}{4}}$ and $\frac{\sqrt{2}}{2}(1+i)$. Taking the real part of the latter and multiplying by the former gives $e^{\frac{\pi}{4}} \cdot \frac{\sqrt{2}}{2}$ which is equivalent to **(B)**.

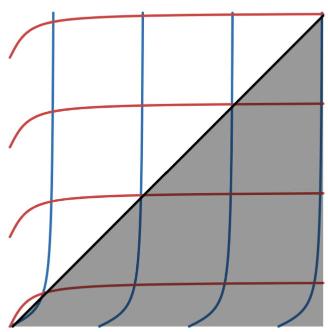
14. Oh no, there's chemistry in the math test! Fortunately, it's just law of cosines. With two sides of length 1.5 making an angle of $88°$, the length of the hypotenuse will be $\sqrt{1.5^2 + 1.5^2 - 2 \cdot 1.5 \cdot \cos 88°} = \boxed{\sqrt{4.50 - 3\cos 88°}}$ **(C)**.

15. Let $u = \tan x$. Multiplying through on both sides yields $(1+u)^2 = \left(1 + \frac{1}{u}\right)^2 = \frac{1}{u^2}(1+u)^2$. Thus we see that $u = \tan x$ is either $+1$ or -1. But $\tan x$ can't be -1, as then we would divide by zero! So only $\tan x = 1$ works, and this occurs at $\frac{\pi}{4}$ and $\frac{5\pi}{4}$. The sum of these solutions is $\boxed{\frac{3\pi}{2}}$ **(A)**.

16. Solving for x yields $x = \frac{1}{2}(2-\sqrt{2})(1+\sqrt{3})$. Approximating $\sqrt{2} \approx 1.4$ and $\sqrt{3} \approx 1.7$ yields $0.81 \approx \boxed{0.8}$ (the actual value is $x \approx 0.8002$). Thus, the answer is **(B)**.

17. Another approximation question. When x is small, $\sin x \approx x$ and $\cos x \approx 1$, so $\cot x \approx \frac{1}{x}$. Since cotangent has a period of $\pi \approx 3.1416$, we are evaluating approximately $\cot(-0.0016) \approx -\frac{1}{0.0016} < -600$. Therefore the answer is $\boxed{-1000}$ **(B)**.

18. Since the period of tangent is π, there is room for 3.5 periods on both the x and y-axis. We can draw a diagram and see there are 10 intersections. However, since $\frac{1}{2}\tan x$ is used, there is an eleventh intersection near $(0,0)$. Thus the answer is $\boxed{11}$ **(C)**.



19. There are some very strong hints to use complex numbers here. Multiplying the second equation by i, if $z = x + yi$, then $z^2 - 2z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Completing the square shows $(z-1)^2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. Therefore, $z = 1 \pm \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$.

The question asks us to find the sum of two tangents, which has $\tan(a+b) = \frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}$. For the x-coordinate, this yields $\frac{2}{1-(1-\frac{3}{4})} = \frac{8}{3}$. For the y-coordinate, since the two values are negatives of each other, their tangent sum will be zero. Hence the answer is just $\boxed{\frac{8}{3}}$ **(C)**.

20. Another question about tangent sums. We can use two strategies to solve this problem. We can either use multiple angle identities, or we can use complex numbers. The tangent of the argument of a complex number $z = x + yi$ is $\frac{y}{x}$, and we know $\arg(a \cdot b) = \arg(a) + \arg(b)$. Therefore, x is equal to the ratio of imaginary to real parts of $(1+i)^3(1+2i)^{-2}(1+3i)$. A quick computation shows this value is $\boxed{\frac{11}{2}}$ **(D)**.

21. The amplitude of $A\cos x + B\sin x$ is $\sqrt{A^2+B^2}$ (this can be proven by converting everything to the same trig function). In this case, we can convert into this form, as $\cos x + 6\sin(x + \frac{\pi}{6}) = 3\sqrt{3}\sin x + 4\cos x$. The amplitude here is $\sqrt{16+27} = \boxed{\sqrt{43}}$. **(D)**.

22. We are solving $\sum_{n=0}^{90} \sin^4(n°)$. First, we see that $\sin^4(x) + \sin^4(90° - x) = \sin^4(x) + \cos^4(x) = (\cos^2 x + \sin^2 x)^2 - 2\cos^2 x \sin^2 x = 1 - \frac{1}{2}\sin^2 2x$. Therefore, $\sum_{n=0}^{90} \sin^4(n°) = \sum_{n=0}^{44}\left(1 - \frac{1}{2}\sin^2(2n°)\right) + \frac{1}{4}$. We can further simplify to $\frac{181}{4} - \frac{1}{2}\sum_{n=0}^{44}\sin^2(2n°) = \frac{181}{4} - \frac{1}{2}\sum_{n=1}^{44}\sin^2(2n°)$. Since $\sin^2(x) + \sin^2(90° - x) = 1$, we get $\frac{181}{4} - \frac{1}{2}22 = \boxed{\frac{137}{4}}$ **(D)**.

23. We have $\sin x + \cos x = \frac{1}{2}$. Squaring shows that $2\sin x \cos x = \frac{1}{4} - 1 = -\frac{3}{4}$. Thus $\sin x \cos x = -\frac{3}{8}$. Using cubes, $\sin^3 x + \cos^3 x = (\sin x + \cos x)^3 - 3(\sin x \cos x)(\sin x + \cos x) = \frac{1}{8} - 3\left(-\frac{3}{8}\right)\frac{1}{2} = \frac{1}{8} + \frac{9}{16} = \frac{11}{16} \Longrightarrow \boxed{27}$ **(D)**.

24. The angle $\theta = \frac{2\pi}{5}$ satisfies the identity $\tan 2\theta = -\tan 3\theta$. If $x = \tan\theta$, we can rewrite this as $\frac{2x}{1-x^2} = -\frac{3x-x^3}{1-3x^2}$. Solving for x gives $2(1-3x^2) = (x^2-3)(1-x^2)$ which can be simplified into $\boxed{x^4+10x^2+5} = 0$. Therefore the answer is **(A)**.

25. The first step is to simplify known quantities. This gives

$$\frac{1+\cot x}{\tan x + \cot x} = \frac{\frac{3}{4} - \sin^2 x}{\cos x - \sin x}.$$

From question 3, $\frac{1}{\tan x + \cot x} = \frac{\sin 2x}{2}$. Factoring $1 + \cot x$ as $\frac{\cos x + \sin x}{\sin x}$, we can multiply both sides through by $(\sin x)(\cos x - \sin x)$ to get

$$\begin{aligned} \frac{1}{2}\sin 2x(\cos^2 x - \sin^2 x) &= \frac{1}{4}(3\sin x - 4\sin^3 x) \\ \frac{1}{4}\sin 4x &= \frac{1}{4}\sin 3x \\ \sin 3x &= \sin 4x. \end{aligned}$$

Such an identity holds if $3x$ and $4x$ are mirrored across $\frac{\pi}{2}$ so for the minimal solution $\frac{3x+4x}{2} = \frac{\pi}{2}$. Hence $x = \boxed{\frac{\pi}{7}}$ **(C)**.

26. We look at the first system. Multiplying both sides by two and subtracting two gives $2\cos^2 x - 1 + 2\cos^2 y - 1 = 0$ which is just two half-angle identities ($\cos 2x = -\cos 2y$). The shape of this graph is infinitely many lines of the form $y = x + \frac{\pi}{2} + \pi k$ and $y = -x + \frac{\pi}{2} + \pi k$. The closest these lines reach to the origin is $\frac{\pi\sqrt{2}}{4}$ which is around 1.1. This value squared is about 1.2, so the actual answer is the ceiling of this which is $\boxed{2}$ **(B)**.

27. Solution 1. We can see pretty clearly that $\angle C$ is acute. Then $\cos\frac{1}{2}\angle C = \frac{4}{\sqrt{17}}$ and $\sin\angle C = \frac{8}{17}$. We notice that it would be useful if the height was 8, and, in fact, it is so – the triangle can be divided into a 6-8-10 and an 8-15-17 triangle. From here, $\cos(\angle A - \angle B) = \cos A\cos B + \sin A \sin B$. $\cos A = \frac{3}{5}, \sin A = \frac{4}{5}$. For B, $\sin B = \frac{84}{85}$ by law of sines and $\cos B = -\frac{13}{85}$ by law of cosines. Therefore, $\cos(\angle A - \angle B) = \frac{297}{425}$ and cosine of half this value is $\boxed{\frac{19}{5\sqrt{17}}}$.

Solution 2. We apply Mollweide's formula, in which $\frac{a+b}{c} = \frac{\cos\frac{1}{2}(\angle A - \angle B)}{\sin\frac{1}{2}\angle C}$. Then with x as the answer, $\frac{19}{5} = x\sqrt{17}$, so the answer is $\boxed{\frac{19}{5\sqrt{17}}}$.

In each case the answer is **(C)**.

28. Let $a_1 = \frac{\sqrt{2}}{2}$ and $a_{n+1} = \sqrt{\frac{1+a_n}{2}}$ for $n \geq 1$. It can be seen that a_n is the n-th term of the product. Therefore, the product is equal to $\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{16}\right)\cdots$

Consider the first k terms of this product ranging from $\cos\left(\frac{\pi}{2\cdot 2}\right)$ to $\cos\left(\frac{\pi}{2\cdot 2^k}\right)$. Multiplying by $\sin\left(\frac{\pi}{2}2^{-k}\right)$ creates a cascade of double-angle identities, resulting in $2^{-k}\sin\left(\frac{\pi}{2}\right) = 2^{-k}$. But when x is small, $\sin x \approx x$, so as k increases the value multiplied approaches $\frac{\pi}{2}2^{-k}$. Since the result is always 2^{-k}, we effectively multiply by $\frac{\pi}{2}$ to get 1. Hence the original product is equal to $\boxed{\frac{2}{\pi}}$ **(C)**.

29. We split the sum up into different sections modulo 5. For $n \equiv 0, 1, 2, 3 \mod 4$, we have $2^n \equiv 1, 2, 4, 3 \mod 5$ respectively. Also, note that using the givens in question 8, $\cos\frac{2\pi}{5} = \cos\frac{8\pi}{5} = \frac{-1+\sqrt{5}}{4}$ and $\cos\frac{4\pi}{5} = \cos\frac{6\pi}{5} = \frac{-1-\sqrt{5}}{4}$.

$$\begin{aligned}\left(1+\frac{1}{16}+\frac{1}{256}+\cdots\right)\cos\frac{2\pi}{5}+\left(\frac{1}{2}+\frac{1}{32}+\cdots\right)\cos\frac{4\pi}{5}+\left(\frac{1}{4}+\frac{1}{64}+\cdots\right)\cos\frac{8\pi}{5}+\left(\frac{1}{8}+\frac{1}{128}+\cdots\right)\cos\frac{6\pi}{5}\\
=\left(\frac{16}{15}+\frac{4}{15}\right)\frac{-1+\sqrt{5}}{4}+\left(\frac{8}{15}+\frac{2}{15}\right)\frac{-1-\sqrt{5}}{4}\\
=\boxed{-\frac{1}{2}+\frac{\sqrt{5}}{6}}\ \textbf{(A)}.\end{aligned}$$

30. No degrees symbol is given to indicate radians are not being used. Per the rules of this test, this means radians are used and the answer is some irrational quantity (about -0.448). Hence the answer is **(E)**.