# Answers

1. $\dfrac{128}{3}$
2. $\dfrac{-1}{2}$
3. $10 + 2\pi^2$
4. $\dfrac{2}{x^3}$
5. $\dfrac{1}{2}$
6. $1 + \ln 2$
7. $\sin 1$
8. $\dfrac{140}{3}$
9. $\dfrac{1}{72}$
10. $4$
11. $\sin 4$
12. $\dfrac{8\pi}{3} - 2\sqrt{3}$
13. $\dfrac{9}{4}$
14. $\dfrac{5}{8}$
15. $\ln 5$
16. $\dfrac{9}{2}$
17. $\sqrt{5}$
18. $1 - \dfrac{2}{\mathrm{e}}$
19. $1$
20. $\pi$
21. $\dfrac{\pi}{2}$
22. $\dfrac{5}{7}$
23. $12$
24. $2025e^7$
25. $2$

# Solutions

**1.** Since $f(2) = 16$, we have

$$f(4) = f(2) + \int_2^4 m(x)\mathrm{d}x = 16 + \int_2^4 (1 + x + x^2)\mathrm{d}x = \frac{128}{3}$$

**2.** Using implicit differentiation, one can check that the derivative of this equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2(x-y)+1}{2(x-y)}$$

We want $2(x-y)+1 = 0$, but $2(x-y) \neq 0$. We see that this implies that $x - y = \boxed{\frac{-1}{2}}$, which is what we want.

**3.** With the substitution $u = x - 2\pi$, one has

$$\int_\pi^{3\pi} f(x)\mathrm{d}x = \int_{-\pi}^{\pi} f(u + 2\pi)\mathrm{d}u = \int_{-\pi}^{\pi} f(u)\mathrm{d}u = 10$$

Therefore

$$\int_\pi^{3\pi} (f(x) + \pi)\mathrm{d}x = \boxed{10 + 2\pi^2}$$

**4.** One can grind through the algebra and obtain the desired answer, or one can notice that the limit is just $f''(x)$. Therefore, the answer is $\boxed{\frac{2}{x^3}}$

**5.** One can interpret this integral as the area of a polar region. Indeed, the curve $r = \dfrac{1}{\sin\theta + \cos\theta}$ represents that line $x + y = 1$. Hence,

$$\frac{1}{2}\int_0^{\pi/4} \frac{1}{(\sin\theta + \cos\theta)^2}\mathrm{d}\theta = \text{Area of } \triangle \text{ with vertices } (0,0), (1,1), (1,0) = \frac{1}{4}$$

Therefore,

$$\int_0^{\pi/4} \frac{1}{(\sin x + \cos x)^2}\mathrm{d}x = 2\frac{1}{4} = \boxed{\frac{1}{2}}$$

**6.** Using arclength formula to find the area of the curved region (the $\ln x$ part) and geometry to find the length of the lines, we see that the perimeter of the region is

$$2 - 1 + \ln 2 + \int_1^2 \sqrt{1 + \left(\frac{1}{x}\right)^2}\,dx = \int_1^2 \frac{\sqrt{1+x^2}}{x}dx + \ln 2 + 1$$

We see that $A = \boxed{1 + \ln 2}$.

**7.** We have

$$\lim_{x\to 1} \frac{\cos x - \cos 1}{1 - x} = -(\cos x)'\Big|_{x=1} = \boxed{\sin 1}$$

**8.**

$$\int_{-3}^{2} 4x^2 dx = \frac{4x^3}{3}\Big|_{-3}^{2} = \boxed{\frac{140}{3}}$$

**9.** Using a modified version of the geometric infinite sum formula $\sum_{n=k}^{\infty} r^n = \frac{r^k}{1-r}$ for $|r| < 1$, one has

$$\sum_{n=2}^{\infty} 3^{-2n} = \sum_{n=2}^{\infty} \frac{1}{9^n} = \frac{(1/9)^2}{1 - 1/9} = \boxed{\frac{1}{72}}$$

**10.** Observe that $|x + |x|| = \begin{cases} 0 & x < 0 \\ 2x & x \geq 0 \end{cases}$ Therefore,

$$\int_{-2}^{2} |x + |x|| = \int_{0}^{2} 2x dx = x^2\Big|_0^2 = \boxed{4}$$

**11.** Using l'Hospital's,

$$\lim_{x\to 4} \frac{x \int_4^x \frac{\sin t}{t} dx}{x - 4} = \lim_{x\to 4} \frac{\int_4^x \frac{\sin t}{t} dx + \frac{x \sin x}{x}}{1} = \lim_{x\to 4} \frac{0 + \sin x}{1} = \boxed{\sin 4}$$

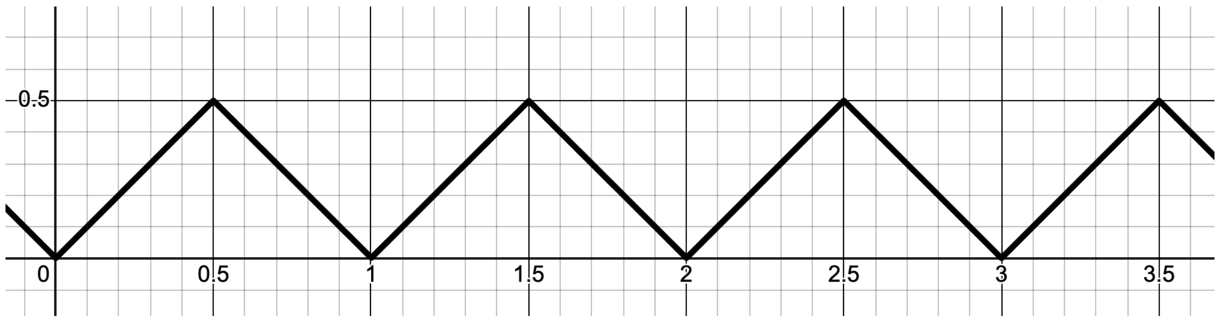
**12.** It is enough to find the area of the requested region in the first quadrant as the region is symmetric with respect to the x-axis. On the first quadrant, $2 = 4\cos\theta$ yields $\theta = \frac{\pi}{3}$. Therefore, the area inside both curves is

$$2A = 2\left(\frac{\pi}{3}\right) + 2\frac{1}{2}\int_{\pi/3}^{\pi/2} (4\cos\theta)^2 d\theta = \frac{2\pi}{3} + 8\int_{\pi/3}^{\pi/2} (1 + \cos 2\theta) d\theta = \boxed{\frac{8\pi}{3} - 2\sqrt{3}}$$

**13.** We may write the equation of the parabola as $y = 4 - (x+1)^2$. It slope is $y' = -2(x+1)$ and since its y-intercept is $(0,3)$, then the tangent at that point is $y = -2x+3$. The x-intercept is $\left(\frac{3}{2}, 0\right)$. The requested area is

$$\frac{1}{2} \cdot 3 \cdot \frac{3}{2} = \boxed{\frac{9}{4}}$$

**14.** Since $x - \lfloor x \rfloor$ is the line $x$ which repeats at every $[0,1]$ and $-x - \lfloor -x \rfloor$ is its reflection along the y-axis and they meet at halfway the unit intervals, then the minimum of both is a bunch of triangles with base 1 and height $\frac{1}{2}$, as pictured below.




Therefore,

$$\int_1^{7/2} \min(x - \lfloor x \rfloor, -x - \lfloor -x \rfloor) \mathrm{d}x = 2\left(\frac{1}{4}\right) + \frac{1}{8} = \boxed{\frac{5}{8}}$$

**15.** We split in the integral in two and exploit symmetry as much as we can:

$$\int_{-2}^{2} \frac{|x| + x}{x^2+1} \mathrm{d}x = \int_{-2}^{2} \frac{|x|}{1+x^2} \mathrm{d}x + 0 = 2\int_0^2 \frac{x}{1+x^2} \mathrm{d}x = \ln(1+x^2)\Big|_0^2 = \boxed{\ln 5}$$

**16.** We have

$$f(2) = f(-1) + \int_{-1}^{2} (1-x) \mathrm{d}x = 3 + \left(x - \frac{x^2}{2}\right)_{-1}^{2} = \boxed{\frac{9}{2}}$$

**17.** The speed is given by

$$\sqrt{(x'(\pi/2))^2 + (y'(\pi/2))^2} = \sqrt{(\cos(\pi/2)^2 + 1)^2 + (1 + \sin(\pi/2)^2} = \sqrt{1^2 + 2^2} = \boxed{\sqrt{5}}$$

**18.** The volume is given by

$$V = \int_0^1 A(x) \mathrm{d}x = \int_0^1 x\mathrm{e}^{-x} \mathrm{d}x = -x\mathrm{e}^{-x}\Big|_0^1 + \int_0^1 \mathrm{e}^{-x} \mathrm{d}x = -\mathrm{e}^{-1} - (\mathrm{e}^{-1} - 1) = \boxed{1 - \frac{2}{\mathrm{e}}}$$

**19.** The convergence of the series implies that $\lim_{n\to\infty} a_n = 0$. Therefore,

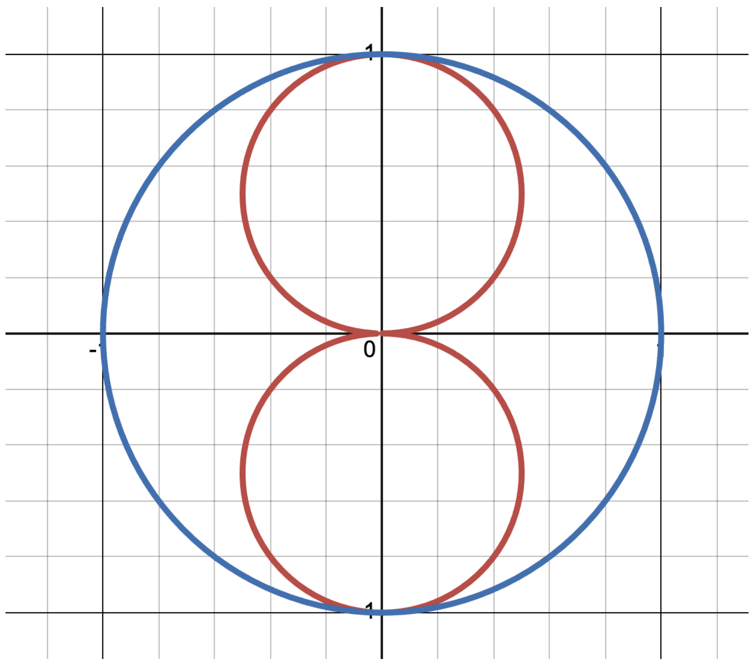
$$\lim_{n\to\infty} \frac{1+a_n}{\cos\left(\frac{1}{n}\right)} = \frac{1+0}{\cos(0)} = \boxed{1}$$

**20.** Recognize the limit as a the limit of Riemann sums:

$$\lim_{n\to\infty} \frac{\pi}{n} \sum_{k=1}^{n} \frac{k\pi}{n} \sin\left(\frac{k\pi}{n}\right) = \int_0^{\pi} x \sin x \mathrm{d}x = -x\cos x\Big|_0^{\pi} + \int_0^{\pi} \cos x \mathrm{d}x = -\pi\cos(\pi) - 0 + 0 = \boxed{\pi}$$

**21.** Observe that $r = |\sin\theta|$ is a curve made of two circles of radius $\frac{1}{2}$ which lie inside the circle $r = -1$.

Therefore, the area bounded by the curves is $\pi - 2\frac{\pi}{4} = \boxed{\frac{\pi}{2}}$




**22.** One may apply the rule of L'Hospital's:

$$\lim_{x\to 1} \frac{1-x^5}{1-x^7} = \lim_{x\to 1} \frac{-5x^4}{-7x^6} = \boxed{\frac{5}{7}}$$

**23.** We know $\sin(x^2) = x^2 - \frac{x^6}{3!} + ....$ Hence, we may expand $G(x)$ in Taylor series around $x = 0$

$$G(x) = \left(\frac{x^2}{2} - \frac{x^6}{3!} + ...\right)\left(x^2 - \frac{x^3}{3} + \frac{x^4}{4} - ...\right) = \frac{x^4}{2} - \frac{x^5}{3} + ...$$

Therefore the coefficient of $x^4$ corresponds to $\frac{G^{(4)}(0)}{4!}$ and this equals $\frac{1}{2}$ from our computation. Therefore, $G^{(4)}(0) = 4!\frac{1}{2} = \boxed{12}$

**24.** Since this is the law of exponential decay, we know the solution is of the form $x(t) = x(0)\mathrm{e}^{-kt}$ where $k = 7$ and $x(0) = 2025$. Therefore, the particular solution is $y(x) = 2025\mathrm{e}^{-7x}$ and $x(-1) = \boxed{2025e^7}$.

**25.** Using chain rule, one has

$$G'(3) = e^{f(3)-2} \cdot f'(3) = \mathrm{e}^{2-2}(2) = \boxed{2}$$

Observe that $f'(3) = 2$ since when $x > 2$, $f(x) = 2(x - 2)$.