Important Instructions For This Test: Throughout this test, ∂V denotes the boundary of V when in integration bounds. Assume that all closed surfaces are positively oriented. Furthermore, x is always the independent variable and y the dependent variable in differential equations. Good luck, have fun, and as always: "NOTA" stands for "None of These Answers is correct."

1. If $f(x, y) = x^2 + xy + y^2$, find ∇f.

(A) $\langle x+y, x+y \rangle$
(B) $\langle 2x+y, x+2y \rangle$
(C) $\langle x+2y, 2x+y \rangle$
(D) $\langle 2x+2y, 2x+2y \rangle$
(E) NOTA

2. Solve the differential equation below for y up to an arbitrary constant C:

$$y' = y(1 - \tan(x))$$

(A) $Cx\cos(x)$ **(B)** $Cx\sin(x)$ **(C)** $Ce^x\cos(x)$ **(D)** $Ce^x\sin(x)$ **(E)** NOTA

3. Evaluate the integral below.

$$\int_0^{\sqrt{\frac{\pi}{2}}} \int_0^x \sin(x^2)\,dy\,dx$$

(A) $\frac{1}{2}$ **(B)** 1 **(C)** $\frac{\pi}{2}$ **(D)** π **(E)** NOTA

4. Find $y(\frac{\pi}{4})$ given that $y(0) = 0$ and y satisfies the differential equation below.

$$\cos(x)y' + \sin(x)y = \cos^2(x)$$

(A) $\frac{\pi}{4\sqrt{2}}$ **(B)** $\frac{\pi}{8\sqrt{2}}$ **(C)** $\frac{\pi}{4\sqrt{2}} + \frac{1}{2\sqrt{2}}$ **(D)** $\frac{\pi}{8\sqrt{2}} + \frac{1}{2\sqrt{2}}$ **(E)** NOTA

5. How many of the following functions satisfy the partial differential equation below?

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

I $f = \cos(x)\cos(y)\cos(z)$
II $f = x^2 + y^2 - 2z^2$
III $f = e^x\cos(y)$
IV $f = x^2 + y^2 + z^2$

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** NOTA

6. Which of the following provides the most general solution to the differential equation below? C and D denote arbitrary constants.

$$y'' - 2y' + 17y = 0$$

(A) $Ce^{4x}\cos(x)$

(B) $Ce^{4x}\cos(x) + De^{4x}\sin(x)$

(C) $Ce^{4x}\cos(x)$

(D) $Ce^{x}\cos(4x) + De^{x}\sin(4x)$

(E) NOTA

7. Find the unit tangent vector $\hat{\mathbf{T}}(t)$ for $\mathbf{r}(t) = \langle e^t\cos(t), e^t\sin(t), \sqrt{2}e^t\rangle$

(A) $e^t\langle\cos(t)-\sin(t), \cos(t)+\sin(t), \sqrt{2}\rangle$

(B) $e^t\langle\cos(t)+\sin(t), \cos(t)-\sin(t), \sqrt{2}\rangle$

(C) $\frac{1}{2}\langle\cos(t)-\sin(t), \cos(t)+\sin(t), \sqrt{2}\rangle$

(D) $\frac{1}{2}\langle\cos(t)+\sin(t), \cos(t)-\sin(t), \sqrt{2}\rangle$

(E) NOTA

8. Find the unit normal vector $\hat{\mathbf{N}}(t)$ for the $\mathbf{r}(t)$ provided in the previous question.

(A) $\frac{1}{2}\langle-\cos(t)-\sin(t), \cos(t)-\sin(t), 0\rangle$

(B) $\frac{1}{\sqrt{2}}\langle-\cos(t)-\sin(t), \cos(t)-\sin(t), 0\rangle$

(C) $e^t\langle-2\sin(t), 2\cos(t), \sqrt{2}\rangle$

(D) $\frac{1}{\sqrt{6}}\langle-2\sin(t), 2\cos(t), \sqrt{2}\rangle$

(E) NOTA

9. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ given $\mathbf{F} = \langle(1+xyz)e^{xyz}, x^2ze^{xyz}, x^2ye^{xyz}\rangle$. C is the curve defined by $\mathbf{r}(t) = \langle\sqrt{1+t^2}, (1+t^2)^{\frac{3}{2}}, 1+t^2\rangle$ where $t : 0 \to 2\sqrt{2}$

(A) $e^{729}-1$ **(B)** $e^{729}-e$ **(C)** $3e^{729}-e$ **(D)** $3e^{729}-e^3$ **(E)** NOTA

10. Find $y(2)$ given that y is subject to the initial conditions $y(1) = 0, y'(1) = 1$, and y satisfies the differential equation

$$x^2y'' - 4xy' + 6y = 0$$

(A) 1 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** NOTA

11. Find the equation of the plane tangent to the surface $x^2 + y^2 + \frac{1}{2}z^2 = 1$ at the point $(\frac{1}{2}, \frac{1}{2}, 1)$

(A) $2x + 2y + z = 3$ **(C)** $x + y + z = 2$ **(E)** NOTA

(B) $4x + 4y + z = 5$ **(D)** $x + y + 2z = 3$

12. Evaluate the integral below:

$$\int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\,dydx$$

(A) $\frac{1}{3}e$ **(B)** $\frac{1}{3}(e-1)$ **(C)** $\frac{1}{6}e$ **(D)** $\frac{1}{6}(e-1)$ **(E)** NOTA

13. If $f(x)$ is the particular solution to the differential equation below (the non-homogenous solution), find $f(0)$.

$$y'' + 2y' + y = \sin(2x)$$

(A) $-\frac{4}{25}$ **(B)** $-\frac{3}{25}$ **(C)** $\frac{3}{25}$ **(D)** $\frac{4}{25}$ **(E)** NOTA

14. If $f(x)$ is the particular solution to the differential equation below, find $f(\frac{1}{2})$.

$$y'' + 4y = \sin(2x)$$

(A) $-\frac{1}{8}\cos(1)$ **(B)** $\frac{1}{8}\cos(1)$ **(C)** $-\frac{1}{8}\sin(1)$ **(D)** $\frac{1}{8}\sin(1)$ **(E)** NOTA

15. Yusuf and Jack are arguing over the correct way to solve the following integral:

$$\oiint_{\partial V} \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2+y^2+z^2)^{\frac{3}{2}}} \cdot d\mathbf{S}$$

Where V is a unit ball centered at the origin. Yusuf applies divergence theorem and gets 0, but Jack just directly solves the flux integral and gets 4π. Who is correct?

(A) Yusuf **(B)** Jack **(C)** Neither **(D)** Need more info **(E)** NOTA

16. Evaluate the line integral below where C is the curve $\frac{x^2}{2}$ from $x = 0$ to $x = 1$.

$$\int_C x^3\,ds$$

(A) $\frac{1}{2}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{2+2\sqrt{2}}{15}$ **(D)** $\frac{2+2\sqrt{2}}{30}$ **(E)** NOTA

17. Find the Laplace Transform of $t^2\sin(t)$ evaluated at $s = 1$. It is helpful to know that $\mathcal{L}\{\sin(t)\} = \frac{1}{s^2+1}$.

(A) $\frac{1}{2}$ **(B)** $\frac{2}{3}$ **(C)** 1 **(D)** 2 **(E)** NOTA

18. Evaluate $\mathcal{L}^{-1}\{s^{-\frac{3}{2}}\}$.

(A) $\sqrt{t}$ **(B)** $2\sqrt{t}$ **(C)** $\sqrt{\frac{t}{\pi}}$ **(D)** $2\sqrt{\frac{t}{\pi}}$ **(E)** NOTA

19. Find the maximum value of $e^{-x^2-y^2-z^2}$ subject to the constraint $x + 2y + 3z = 1$.

(A) $e^{-\frac{1}{3}}$ **(B)** $e^{-\frac{1}{10}}$ **(C)** $e^{-\frac{1}{14}}$ **(D)** $e^{-\frac{1}{21}}$ **(E)** NOTA

The following information may be useful for 20-21: ∇^2 is known as the Laplacian, and it is equivalent to $\nabla \cdot \nabla$. In 3-dimensional cartesian coordinates,

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

The partial differential equation $\nabla^2 f = 0$ is known as Laplace's equation, and it has important applications in physics and math.

20. Let f be some smooth function that satisfies $\nabla^2 f = 1$. Furthermore, let $f_{ave}(r)$ denote the average value of f over some sphere of radius r centered at the origin. Given that

$$\frac{df_{ave}}{dr} = \frac{1}{4\pi r^2} \oiint_{\partial B} \nabla f \cdot d\mathbf{S}$$

where B is a ball of radius r centered at the origin, find an expression for $f_{ave}(r)$ up to an arbitrary constant C.

(A) $\frac{r^2}{6} + C$ **(B)** $\frac{r^2}{2} + C$ **(C)** $\frac{r}{3} + C$ **(D)** $\frac{r}{2} + C$ **(E)** NOTA

21. Suppose f satisfies Laplace's equation in 3d with the boundary condition that $f(x, y, z) = 1 - z^2$ on the unit sphere centered at the origin. What is $f(0, 0, 0)$?

(A) 0 **(B)** $\frac{1}{3}$ **(C)** $\frac{2}{3}$ **(D)** 1 **(E)** NOTA

22. Jun is blowing a bubble that is in the shape of the surface defined by V: $x^2 + y^2 + z^2 = 2zt + 1$ for $z \geq 0$ where t is time. His air flow can be described as the vector field $\mathbf{A} = \langle 0, -e^{-y-z}, e^{-y-z} \rangle$. Let

$$\Phi = \iint_V \mathbf{A} \cdot d\mathbf{S}$$

What is $\frac{d\Phi}{dt}$?

(A) 0 **(B)** $\frac{8}{3}\pi t$ **(C)** $\frac{2}{3}\pi t^2$ **(D)** $\frac{4}{3}\pi t^2$ **(E)** NOTA

23. Let $y(x)$ be a function such that it minimizes the integral below:

$$\int_0^{\frac{\pi}{2}} \sqrt{y^2 + \left(\frac{dy}{dx}\right)^2}\, dx$$

with boundary conditions $y(0) = 1$ and $y(\frac{\pi}{2}) = 1$. Find $y(\frac{\pi}{3})$.

(A) $\frac{1}{2}$ **(B)** $\frac{\sqrt{3}}{2}$ **(C)** $\sqrt{3}+1$ **(D)** $\sqrt{3}-1$ **(E)** NOTA

24. What is the following equal to?

$$\lim_{n\to\infty} \left(\frac{1}{2}\ln(n) + \sum_{k=1}^{2n} (-1)^{k+1} \ln(k) \right)$$

(A) $-\frac{1}{2}\ln(\pi)$ **(B)** $\frac{1}{2}\ln(\pi)$ **(C)** $\ln(\pi)$ **(D)** ∞ **(E)** NOTA

25. The vector area **a**, of a surface A is defined as

$$\mathbf{a} = \int_A d\mathbf{S}$$

Find the vector area of the surface $x^2 + y^2 + z^2 = 1$ restricted to $x + y + z > 1$.

(A) $\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$ **(C)** $\langle \frac{2\pi}{3\sqrt{3}}, \frac{2\pi}{3\sqrt{3}}, \frac{2\pi}{3\sqrt{3}} \rangle$ **(E)** NOTA

(B) $\langle \frac{\pi}{3\sqrt{3}}, \frac{\pi}{3\sqrt{3}}, \frac{\pi}{3\sqrt{3}} \rangle$ **(D)** $\langle \frac{\pi}{\sqrt{3}}, \frac{\pi}{\sqrt{3}}, \frac{\pi}{\sqrt{3}} \rangle$

26. Evaluate the following integral:

$$\iiiint_{\mathbb{R}^4} e^{\mathbf{x}^T \mathcal{A} \mathbf{x}} dx_1 dx_2 dx_3 dx_4$$

where $\mathbf{x} = \langle x_1, x_2, x_3, x_4 \rangle$ and

$$\mathcal{A} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -14 & 12 \\ 0 & 0 & 12 & -21 \end{pmatrix}$$

Note: A 1-by-1 matrix is a scalar, e.g. $e^{[-x^2]} = e^{-x^2}$.

(A) $14\sqrt{3}\pi^2$ **(B)** $10\sqrt{3}\pi^2$ **(C)** $\frac{\pi^2}{14\sqrt{3}}$ **(D)** $\frac{\pi^2}{10\sqrt{3}}$ **(E)** NOTA

The following information may be useful for 27-28:

The Bessel function is a special function that has many important applications in physics and math. It is a solution to the differential equation below:

$$x^2y'' + xy' + (n^2 - x^2)y = 0$$

and is equal, for integer n, to

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x}{2}\right)^{2k+n}$$

27. Evaluate the integral below in terms of Bessel functions.

$$\int_0^{\pi} \cos(2\sin(x))dx$$

(A) $\pi J_2(0)$ **(B)** $\pi J_0(1)$ **(C)** $\pi J_1(0)$ **(D)** $\pi J_0(2)$ **(E)** NOTA

28. Evaluate the integral below

$$\int_0^{\infty} J_0(x)e^{-x}dx$$

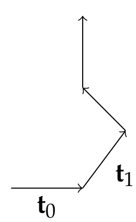
(A) 1 **(B)** $\frac{1}{\sqrt{2}}$ **(C)** π **(D)** $\frac{\pi}{\sqrt{2}}$ **(E)** NOTA

29. Find $y(-1)$ given that $y(0) = 0$, $y(1) = 1$, and y satisfies the differential equation

$$y'' + 2xy' + (x^2 + 1)y = 0$$

(A) $-\sqrt{e}$ **(B)** -1 **(C)** 1 **(D)** $\sqrt{e}$ **(E)** NOTA

30. Consider a chain of connected unit vectors $\{\mathbf{t}_0 \ldots \mathbf{t}_l\}$.



Where $\mathbf{t}_i \cdot \mathbf{t}_{i+1} = \cos(\theta_i)$. The probability of any given configuration is proportional to $e^{\mathcal{H}}$ where

$$\mathcal{H} = \sum_{i=1}^{l-1} \cos(\theta_i)$$

Let $1 < n < m < l$. What is the expected value of $\mathbf{t}_n \cdot \mathbf{t}_m$ equal to in terms of m and n? C denotes some constant that is not necessarily expressible analytically.

(A) C **(B)** C^{m-n} **(C)** $\frac{C}{m-n}$ **(D)** $\frac{C}{(m-n)^2}$ **(E)** NOTA