Answer Key:

1. **B**
2. **C**
3. **A**
4. **A**
5. **B**
6. **D**
7. **C**
8. **B**
9. **C**
10. **C**
11. **C**
12. **B**
13. **A**
14. **A**
15. **B**
16. **C**
17. **A**
18. **D**
19. **C**
20. **A**
21. **C**
22. **A**
23. **D**
24. **A**
25. **C**
26. **D**
27. **D**
28. **B**
29. **B**
30. **B**

Solutions:

1. $\boxed{\textbf{B}}$: Recall that $\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$. This is simply $< 2x + y, x + 2y >$.

2. $\boxed{\textbf{C}}$: This is a separable differential equation:

$$\int \frac{dy}{y} = \int (1 - \tan(x))dx$$

$$\ln(y) = (x - \ln(\sec(x))) + C$$

$$\ln(y) = (x + \ln(\cos(x))) + C$$

$$y = Ce^x \cos(x)$$

3. $\boxed{\textbf{A}}$:

$$\int_0^{\sqrt{\frac{\pi}{2}}} x \sin(x^2)dx$$

We can u sub $u = x^2$:

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(u)du = \frac{1}{2}$$

4. $\boxed{\textbf{A}}$: Since this is a first order linear differential equation, we can use integrating factor to solve it.

$$\mu(x)y' + \mu(x)\tan(x)y = \mu(x)\cos(x)$$

$$\mu'(x) = \mu(x)\tan(x)$$

$$\frac{\mu'(x)}{\mu(x)} = \tan(x)$$

$$\ln(\mu(x)) = \ln|\sec(x)|$$

$$\mu(x) = \sec(x)$$

$$(\sec(x)y)' = 1$$

$$y = x\cos(x) + C\cos(x)$$

Plugging in the initial conditions, we get that $y = x\cos(x)$. Therefore, $y(\frac{\pi}{4}) = \frac{\pi}{4\sqrt{2}}$

5. $\boxed{\text{B}}$: We just need to plug in each of the functions to see which one satisfies the differential equation. You will find that II. and III. do:

$$\text{II:}\ \ 2+2-4=0$$

$$\text{III:}\ \ e^x\cos(x)-e^x\cos(x)=0$$

But, I. and IV. do not:

$$\text{I:}\ \ -\cos(x)\cos(y)\cos(z)-\cos(x)\cos(y)\cos(z)-\cos(x)\cos(y)\cos(z)\neq 0$$

$$\text{IV:}\ \ 2+2+2\neq 0$$

Therefore, 2 of them satisfy the partial differential equation.

6. $\boxed{\text{D}}$: We can solve the characteristic equation $r^2+2r+17$ to get that $r=1+4i$. Therefore, $y=Ce^x\cos(4x)+De^x\sin(4x)$.

7. $\boxed{\text{C}}$: $\hat{\mathbf{T}}(t)=\frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}$.

$$\mathbf{r}'(t)=e^t<\cos(t)-\sin(t),\cos(t)+\sin(t),\sqrt{2}>$$

$$||\mathbf{r}'(t)||=2e^t$$

Therefore, the correct answer is C.

8. $\boxed{\text{B}}$: $\hat{\mathbf{N}}(t)=\frac{\hat{\mathbf{T}}'(t)}{||\hat{\mathbf{T}}'(t)||}$.

$$\hat{\mathbf{T}}'(t)=\frac{1}{2}<-\cos(t)-\sin(t),\cos(t)-\sin(t),0>$$

$$||\hat{\mathbf{T}}'(t)||=\frac{1}{\sqrt{2}}$$

Therefore, the correct answer is B.

9. $\boxed{\text{C}}$: $\mathbf{F}$ is conservative. A valid potential function is $f=xe^{xyz}$. $f(\mathbf{r}(2\sqrt{2}))-f(\mathbf{r}(0))=3e^{729}-e$

10. $\boxed{\text{C}}$: This is a Cauchy-Euler equation. These differential equations have ansatzes of x^r. Plugging this in and solving for r,

$$r(r-1)x^r-4rx^r+6x^r=0$$

$$r^2-5r+6=0\implies r=2,3$$

Therefore, $y=c_1x^2+c_2x^3$. Solving for the initial conditions and plugging in $x=2$ gives us $y(2)=4$.

11. $\boxed{\text{C}}$: The gradient at this point is $< 1, 1, 1, >$, so we know this is the normal vector. Then we know it passes through $(\frac{1}{2}, \frac{1}{2}, 1)$, so the equation of the plane is $(x - \frac{1}{2}) + (y - \frac{1}{2}) + (z - 1) = 0 \implies x + y + z = 2$.

12. $\boxed{\text{B}}$: In order to solve this integral, we need to switch the order of integration:

$$\int_0^1 \int_0^{y^2} e^{y^3} dx dy$$

$$\int_0^1 y^2 e^{y^3} dy$$

If we do u-sub with $u = y^3$, we can see that the value of the integral is $\frac{1}{3}(e - 1)$.

13. $\boxed{\text{A}}$: The ansatz is $A \sin(2x) + B \cos(2x)$. Plugging this into the differential equation and equating coefficients, we get

$$\begin{cases} -4A - 4B + A = 1 \\ -4B + 4A + B = 0 \end{cases}$$

Solving for A and B, we get $A = -\frac{3}{25}$ and $B = -\frac{4}{25}$. This means our particular solution is $f(x) = -\frac{3}{25}\sin(2x) - \frac{4}{25}\cos(2x)$ which gives us $f(0) = -\frac{4}{25}$.

14. $\boxed{\text{A}}$: The ansatz for this is $Ax \sin(2x) + Bx \cos(2x)$. Plugging this into the differential equation and simplifying gives

$$4A \cos(2x) - 4B \sin(2x) = \sin(2x)$$

$A = 0, B = -\frac{1}{4}$. Therefore, the particular solution is $f(x) = -\frac{1}{4}x\cos(2x)$. $f(\frac{1}{2}) = -\frac{1}{8}\cos(1)$.

15. $\boxed{\text{B}}$: When you apply divergence theorem, you integrate over a singularity at the origin, but this problem is avoided when directly solving the surface integral, so Jack is right. In fact, the divergence of the integrand is not 0, but rather $4\pi\delta(x)\delta(y)\delta(z)$. Where δ is the dirac delta function.

16. $\boxed{\text{C}}$:

$$\int_0^1 x^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$\int_0^1 x^3 \sqrt{1 + x^2} dx$$

If you sub $u = 1 + x^2$, the integral evaluates to $\frac{2+2\sqrt{2}}{15}$.

17. $\boxed{\text{A}}$: This is equivalent to taking the second derivative with respect to s of the laplace transform of $\frac{1}{s^2+1}$. The derivative is equal to $\frac{8x^2}{(1+x^2)^3} - \frac{2}{(1+x^2)^2}$. Evaluating this at $s = 1$ gives $\frac{1}{2}$.

18. $\boxed{\textbf{D}}$: $\mathcal{L}\{\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}}$:

$$\int_0^\infty \sqrt{t}e^{-st}dt$$

Let $u = \sqrt{t}$:

$$\int_0^\infty 2u^2e^{-su^2}du = -2\frac{\partial}{\partial s}\int_0^\infty e^{-su^2} = -\frac{\partial}{\partial s}\sqrt{\frac{\pi}{s}} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

Therefore, $\mathcal{L}^{-1}\{s^{-3/2}\} = 2\sqrt{\frac{\pi}{t}}$

19. $\boxed{\textbf{C}}$: Maximimizing $e^{-x^2-y^2-z^2}$ is equivalent to minimizing $x^2+y^2+z^2$. From here we can use Lagrange multipliers to solve for the minimum with objective function $f = x^2+y^2+z^2$ and constraint function $g = x+2y+3z-1$. We need to solve the system $\nabla f = \lambda \nabla g$, with $g = 0$:

$$\begin{cases} 2x = \lambda \\ 2y = 2\lambda \\ 2z = 3\lambda \\ x+2y+3z = 1 \end{cases}$$

Therefore, $\lambda = \frac{1}{7}$, and $x = \frac{1}{14}, y = \frac{1}{7}, z = \frac{3}{14}$. The minimum value is therefore $e^{-\frac{1}{14}}$

20. $\boxed{\textbf{A}}$: We can just apply divergence theorem to the integral. It is then equal to

$$\frac{df_{ave}}{dr} = \frac{1}{4\pi r^2}\iiint_B \nabla^2 f dV$$

$$\frac{df_{ave}}{dr} = \frac{1}{4\pi r^2}\iiint_B dV$$

This integral is just the volume of the ball

$$\frac{df_{ave}}{dr} = \frac{1}{4\pi r^2}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{df_{ave}}{dr} = \frac{r}{3}$$

$$f_{ave} = \frac{r^2}{6} + C$$

21. $\boxed{\textbf{C}}$: The general idea is to take $\lim_{r\to 0} f_{ave}(r) = f(0,0,0)$. If we repeat question 20 for $\nabla^2 f = 0$, we see that $f_{ave} = C$. This means that regardless of the radius of the sphere we are averaging over, f_{ave} is constant. Therefore, we can just calculate f_{ave} over the boundary to get $f(0,0,0)$. This is

$$\frac{1}{4\pi}\iint_{\partial B}(1-z^2)dS$$

Since $r = 1$ over the surface, $z = \cos(\phi)$ and the Jacobian is just $\sin(\phi)$ when converting to spherical coordinates.

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}(1-\cos^2(\phi))\sin(\phi)d\phi d\theta = \frac{2}{3}$$

22. $\boxed{\text{A}}$: This seems like a rather nasty quantity to calculate, so there might be easier way to do it. We can't apply divergence theorem since the surface isn't closed, so we can try applying Stoke's theorem:

$$\iint_V \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_{\partial V} \mathbf{F} \cdot d\mathbf{r}$$

We now examine our integration boundaries. If we complete the square, we get $x^2 + y^2 + (z-t)^2 = t^2 + 1$. This means the bubble is cut off by the $z = 0$ plane since the radius is slightly bigger than the distance from the center to $z = 0$. Plugging in $z = 0$, we see that the boundary is $x^2 + y^2 = 1$. Therefore the boundary is independent of time.

In order to apply Stoke's theorem, there needs to be some $\mathbf{B}$ such that $\nabla \times \mathbf{B} = \mathbf{A}$. We can see that this is true since $\nabla \cdot \mathbf{A} = 0$, and zero divergence implies there exists some $\mathbf{B}$. Our integral becomes

$$\int_{\partial V} \mathbf{B} \cdot d\mathbf{r}$$

But this is completely independent of time, so $\frac{d\Phi}{dt} = 0$.

23. $\boxed{\text{D}}$: The integral is arclength in polar coordinates: $\int \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$, so we replace y with r for the rest of the problem. The shortest distance between two points is a line, so we want a line in polar coordinates that passes through the provided points. This is the line $x + y = 1$, so $r\cos(\theta) + r\sin(\theta) = 1$, and $r = \frac{1}{\cos(\theta)+\sin(\theta)}$. $r(\frac{\pi}{3}) = \sqrt{3} - 1$

24. $\boxed{\text{A}}$: The sum can be simplified:

$$\lim_{n\to\infty} \frac{1}{2}\ln(n) + \ln\left(\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}\right)$$

$$\lim_{n\to\infty} \frac{1}{2}\ln(n) + \ln\left(\frac{(2n-1)!!}{(2n)!!}\right)$$

Multiply top and bottom by $(2n)!!$

$$\lim_{n\to\infty} \frac{1}{2}\ln(n) + \ln\left(\frac{(2n)!}{((2n)!!)^2}\right)$$

$(2n)!! = 2^n \cdot n!$

$$\lim_{n\to\infty} \frac{1}{2}\ln(n) + \ln\left(\frac{(2n)!}{4^n(n!)^2}\right)$$

Applying Stirling's approximation

$$\lim_{n\to\infty} \frac{1}{2}\ln(n) + \ln\left(\frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{4^n(2\pi n)\left(\frac{n}{e}\right)^{2n}}\right)$$

Simplifying,

$$\lim_{n\to\infty} \frac{1}{2}\ln(n) + \ln\left(\frac{1}{\sqrt{\pi n}}\right) = -\frac{1}{2}\ln(\pi)$$

25. $\boxed{\text{C}}$: The vector area of any closed surface is zero by the divergence theorem. Suppose there were two surfaces with vector areas $\mathbf{a}_1$ and $\mathbf{a}_2$ that shared the same boundary but were not equal to each other. Then, joining the two surfaces would form a closed surface so $\int d\mathbf{S} = \mathbf{a}_1 - \mathbf{a}_2 = 0$ which implies $\mathbf{a}_1 = \mathbf{a}_2$. Therefore, we just need to find some surface with the same boundary. The intersection of the sphere and the plane is just a circle, so we can take the surface to be the disk with that circle as the boundary.

$$\int d\mathbf{S} = \int \hat{\mathbf{n}} dS$$

But, $\hat{\mathbf{n}}$ is constant over the disk since its a subset of the plane and it equals $< \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} >$. Therefore, our integral simplifies:

$$\hat{\mathbf{n}} \int dS$$

The integral is the area of the disk. We can draw a triangle with the hypotenuse being the radius of the sphere and the other leg being the distance from the center of the sphere to the plane. The other leg will the radius of the disk. This gives an area of $\frac{2\pi}{3}$. Multiplying this by $\hat{\mathbf{n}}$ gives us a final answer of $< \frac{2\pi}{3\sqrt{3}}, \frac{2\pi}{3\sqrt{3}}, \frac{2\pi}{3\sqrt{3}} >$.

26. $\boxed{\text{D}}$: If we expand $\mathbf{x}^T \mathcal{A} \mathbf{x}$, we get

$$-x_1^2 - 2x_2^2 - 14x_3^2 + 24x_3x_4 - 21x_4^2$$

We can split this into a product of 3 integrals:

$$\int_{\mathbb{R}} e^{-x_1^2} dx_1 \int_{\mathbb{R}} e^{-2x_2^2} dx_2 \int_{\mathbb{R}^2} e^{-14x_3^2 + 24x_3x_4 - 21x_4^2} dx_3 dx_4$$

The first two are standard Gaussian integrals, but the third integral is messier. We can turn it back into a matrix:

$$\int_{\mathbb{R}^2} e^{\mathbf{x}^T \mathcal{B} \mathbf{x}} dx_3 dx_4$$

where

$$\mathcal{B} = \begin{pmatrix} -14 & 12 \\ 12 & -21 \end{pmatrix}$$

We can now diagonalize the matrix:

$$\mathcal{B} = O^T D O = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}^T \begin{pmatrix} -30 & 0 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

Therefore, the exponent can be rewritten as $(O\mathbf{x})^T D (O\mathbf{x})$. We can now do a $\mathbf{u}$ sub with $\mathbf{u} = O\mathbf{x}$. Since O is an orthogonal matrix, the Jacobian will be 1, but also just evaluating the Jacobian is straightforward. The integral becomes

$$\int_{\mathbb{R}^2} e^{-30u_1^2 - 5u_2^2} du_1 du_2$$

This is again just two standard Gaussian integrals. Multiplying them all together gives $\frac{\pi^2}{10\sqrt{3}}$.

27. $\boxed{\textbf{D}}$: We can taylor expand cos:

$$\int_0^\pi \sum_{k=0}^\infty \frac{(-4)^k \sin^{2k}(x)}{(2k)!} dx$$

$$\sum_{k=0}^\infty \frac{(-4)^k}{(2k)!} \int_0^\pi \sin^{2k}(x) dx$$

The fastest way to do the integral is to know the Wallis Product expression for this integral. However, we derive it here:

$$I(k) = \int_0^\pi \sin^{2k}(x) dx$$

$$I(k) = -\cos(x)\sin^{2k-1}(x)\Big|_0^\pi - \int_0^\pi (2k-1)\sin^{2k-2}(x)\cos^2(x) dx$$

$$I(k) = (2k-1)\int_0^\pi sin^{2k}(x) - \sin^{2k-2}(x) dx$$

$$I(k) = \frac{2k-1}{2k} I(k-1)$$

$$I(0) = \pi$$

$$I(k) = \frac{(2k-1)!!}{(2k)!!}\pi$$

Plugging this into our original sum,

$$\pi \sum_{k=0}^\infty \frac{(-4)^k}{(2k)!} \frac{(2k-1)!!}{(2k)!!} dx$$

$$\pi \sum_{k=0}^\infty \frac{(-4)^k}{((2k)!!)^2} \pi dx$$

$$\pi \sum_{k=0}^\infty \frac{(-4)^k}{(k!2^k)^2}$$

$$\pi \sum_{k=0}^\infty \frac{(-1)^k}{(k!)^2}$$

From here, its straightforward to see that this is $\pi J_0(2)$ when we compare it to the series provided in the information above the problem.

28. $\boxed{\textbf{B}}$: We will take the Laplace transform of $J_0(x)$ and then plug in $s = 1$ since this is equivalent to the integral.

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}$$

from the sum provided. We can then take the Laplace transform of J_0:

$$\mathcal{L}\left\{J_0(x)\right\} = \mathcal{L}\left\{\sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}\right\}$$

By the linearity of the Laplace transform we can distribute the transform into the sum.

$$\mathcal{L}\left\{\sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}\right\} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{1}{2}\right)^{2k} \mathcal{L}\{(x)^{2k}\}$$

$$\mathcal{L}\{(x)^{2k}\} = \frac{(2k)!}{s^{2k+1}}$$

So, the sum reduces to

$$\frac{1}{s} \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{(k!)^2} \left(\frac{1}{2}\right)^{2k} \left(\frac{1}{s}\right)^{2k}$$

If we write out the first few terms of this series:

$$\frac{1}{s}\left(1 - \frac{1}{2}\frac{1}{s^2} + \frac{1 \cdot 3}{2 \cdot 4}\left(\frac{1}{s^2}\right)^2 + \dots\right)$$

We can deduce that this seems familiar to the fractional binomial theorem. It is equal to

$$\frac{1}{s}(1 + \frac{1}{s^2})^{-\frac{1}{2}}$$

Therefore,

$$\mathcal{L}\{J_0(x)\} = \frac{1}{\sqrt{s^2 + 1}}$$

To find the integral in question, we plug in $s = 1$ to get $\frac{1}{\sqrt{2}}$

29. $\boxed{\textbf{B}}$: The differential equation seems familiar to a perfect square binomial, so we can test various squared linear transforms to see which one reproduces the differential equation. A little bit of trial and error will recover that

$$(\frac{d}{dx} + x)(\frac{d}{dx} + x)y = 0$$

is equivalent to the equation provided.

Therefore, y must satisfy $y' + xy = 0$ which gives us our first solution $y = Ce^{-\frac{1}{2}x^2}$. Since this is a second order linear equation, we know that there are two linearly independent solutions, so we can use reduction of order to recover the second solution. We assume that the $y = Cve^{-\frac{1}{2}x^2}$ where v is a function of x. Plugging this into the differential equation (and dividng by C) gives us

$$v''e^{-\frac{1}{2}x^2} - 2xv'e^{-\frac{1}{2}x^2} - ve^{-\frac{1}{2}x^2} + x^2ve^{-\frac{1}{2}x^2} + 2x(v'e^{-\frac{1}{2}x^2} - xve^{-\frac{1}{2}x^2}) + (x^2 + 1)(ve^{-\frac{1}{2}x^2}) = 0$$

Which reduces to

$$v''e^{-\frac{1}{2}x^2} = 0$$

Therefore,

$$v = Cx + D$$

and the general solution to the differential equation is

$$y = (Cx + D)e^{-\frac{1}{2}x^2}$$

Plugging in the initial conditions yields $C = \sqrt{e}$ and $D = 0$. Therefore, $y(-1) = -1$.

30. $\boxed{\textbf{B}}$: Multiple integrals are abbreviated as a single integral throughout this solution.

$$\langle \mathbf{t}_n \cdot \mathbf{t}_m \rangle = \frac{\int (\mathbf{t}_n \cdot \mathbf{t}_m) e^{\mathcal{H}} d\theta_1 \cdots d\theta_l}{\int e^{\mathcal{H}} d\theta_1 \cdots d\theta_l}$$

Where the bounds for each θ_i is from 0 to 2π.

The bottom integral is equivalent to the product of l identical integrals:

$$\int_{\{0<\theta_i<2\pi\ \forall i\}} e^{\cos(\theta_1)+\cdots+\cos(\theta_l)} d\theta_1 \ldots d\theta_l = \left(\int_0^{2\pi} e^{\cos(\theta) d\theta}\right)^l$$

We can always pick θ_i such that $\mathbf{t}_n \cdot \mathbf{t}_m = \cos(\theta_n + \cdots + \theta_{m-1}) = \Re\{e^{i\theta_n + \cdots + i\theta_{m-1}}\}$. If we plug this into our top integral, we will have $l - n + m$ factors of $\int_0^{2\pi} e^{\cos(\theta)} d\theta$ and $m - n$ factors of $\int_0^{2\pi} e^{i\theta + \cos(\theta)} d\theta$. This simplifies our quotient into

$$\Re\left\{ \left(\frac{\int_0^{2\pi} e^{i\theta+\cos(\theta)} d\theta}{\int_0^{2\pi} e^{\cos(\theta)} d\theta} \right)^{m-n} \right\}$$

But, $\int_0^{2\pi} e^{i\theta+\cos(\theta)} d\theta$ is real because

$$\int_0^{2\pi} e^{i\theta+\cos(\theta)} d\theta = \int_0^{2\pi} \cos(\theta) e^{\cos(\theta)} d\theta + i \int_0^{2\pi} \sin(\theta) e^{\cos(\theta)} d\theta$$

Doing a bounds shift, $\theta \to \theta - \pi$:

$$-\int_{-\pi}^{\pi} \cos(\theta) e^{\cos(\theta)} d\theta - i \int_{-\pi}^{\pi} \sin(\theta) e^{\cos(\theta)} d\theta = -\int_{-\pi}^{\pi} \cos(\theta) e^{\cos(\theta)} d\theta$$

$\int_{-\pi}^{\pi} \sin(\theta) e^{\cos(\theta)} = 0$ since it is an odd integral over symmetric bounds. The remaining integral is real. Evaluating the real part of everything in $\Re\{\}$, we are left with

$$\left(\frac{\int_0^{2\pi} e^{i\theta+\cos(\theta)} d\theta}{\int_0^{2\pi} e^{\cos(\theta)} d\theta} \right)^{m-n} = C^{m-n}$$

For some constant $C \approx 0.44$.