All uppercase letter variables are positive integers unless otherwise stated. All fractions containing uppercase letter variables are in lowest terms. NOTA means "None of the Above."

~~~~~~~~ *Good luck, and have fun!* ~~~~~~~~

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1) I love LADDERS!!! Find the number of permutations of the letters in the word `LADDERS`.

A) 720 B) 1440 C) 2520 D) 5040 E) NOTA

2) I begin climbing up a 17-foot ladder that reaches a height of 15 feet. If my height increases at a rate of 3 feet per second, find the rate at which I climb the ladder (in feet per second).

A) $\frac{24}{17}$ B) $\frac{45}{17}$ C) $\frac{17}{5}$ D) $\frac{51}{8}$ E) NOTA

3) I climb the ladder to poke a hole in a full 8-foot tall can of lemonade whose base is 8 feet off the ground. The can is open at the top, so Torricelli's law applies, and the velocity of lemonade coming out of the whole I poke is $\sqrt{2gh}$, where h is the distance from the top of the can to the hole and g is the acceleration due to gravity, 32 feet per second squared. Find the maximum velocity of lemonade exiting the can out of the hole created by my poking the can.

A) $8\sqrt{2}$ B) 16 C) $16\sqrt{2}$ D) 32 E) NOTA

4) My friend tosses me another can of lemonade, and I poke a hole in it so that a jet of lemonade comes out of it at a rate of 8 feet per second. If I'm holding the can so the hole is 24 feet above the ground and the lemonade comes out at an angle θ above the horizontal, find the value of θ that maximizes the horizontal distance the lemonade travels before hitting the ground.

A) $\arcsin \frac{1}{\sqrt{26}}$ B) $\arcsin \frac{1}{2\sqrt{6}}$ C) $\frac{\pi}{12}$ D) $\arcsin \frac{1}{3}$ E) NOTA

5) My friend Frank has a $24\sqrt{3}$-foot long ladder. He carries it through a 8-foot wide hallway, turning around a right angle corner into a hallway with width w. If the minimum possible value of w can be expressed as N^q for rational q and N minimized, find Nq.

A) 8 B) 9 C) 22.5 D) 27 E) NOTA

6) "Who wants to see the Ladders professor go *HIGHER*!" I yell as I climb up the steepest part of the ladder that looks like the graph of $y = 9 - 15x + 36x^2 + 2x^3 - x^4$ in the first quadrant. Find the x-coordinate of my location when I say this, given that the steepest part of the ladder is the point in the first quadrant where the tangent line to the graph has greatest slope.

A) 1.5 B) 2 C) 2.5 D) 3 E) NOTA

7) I fall from my ladder, having been 16 feet above the ground. I was wearing protection, so I bounce back to a height of 2 feet, and subsequent bounces are always to a height of $\frac{1}{8}$ that of the previous bounce. If the total vertical distance I travel (in feet) is $\frac{A}{B}$, find $A + B$.

A) 135 B) 151 C) 263 D) 2191 E) NOTA
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8) Welcome to the section about ADDERS!!! Consider the following procedure for adding any two integers a and b.

1: While $b \neq 0$:

  a: Subtract 1 from b.

  b: Add 1 to a.

2: Return a.

Which of the following individual steps, if added, would resolve all problems with this procedure?

A) Converting a and b to binary B) Ensuring that $a > b$ before the while loop
C) Only performing step 1 if b is positive D) Returning b if $a = 0$
E) NOTA

(This concludes the section about adders.)

For Questions $9 - 13$, you may use the following information. The partial derivative of a multivariable function is the derivative of the function taken with respect to exactly one variable, treating all other variables as constants. A partial derivative of f take with respect to x can be denoted $D_x f$, $\frac{\partial f}{\partial x}$, or f_x. A second partial derivative can be taken with respect to the same or a different variable; for example, two second partial derivatives of f are $D_{xx} f$ and $D_{xy} f$. For example, for the function $g(x, y) = 2x^3 - 7xy^4$, $g_x = 6x^2y^2 - 7y^4$ and $g_{xy} = 12x^2y - 28y^3$. Evaluated at a point, $g_x(1, 2) = 6(1)^2(2)^2 - 7(2)^4 = -88$ and $g_{xy}(1, 2) = 12(1)^2(2) - 28(2)^3 = -200$.

9) The graph of a function in space can have a relative extremum or a saddle point when all partial derivatives of the function evaluated at the point have value 0. The graph of the relation $z = x^2 + 2xy + 2y^2 + 4x + 3y + 5$ is an elliptic paraboloid and has an absolute minimum of $\frac{A}{B}$. Find $A + B$.

A) 7 B) 11 C) 13 D) 17 E) NOTA

10) The gradient of a function ∇f is a vector consisting of all of its partial derivatives. The directional derivative of a function is the dot product of the gradient of the function with a unit vector pointing towards the desired point. A thin metal plate in the xy-plane has temperature $T(x, y) = 100 - (x + 2y)^2$. If the rate of change of temperature at the point $(-4, 1)$ in the direction of $(8, 6)$ is equal to $\frac{A}{B}$, find $A + B$.

A) 9 B) 49 C) 101 D) 119 E) NOTA

11) $u(x, t)$ is a solution to the heat equation, $u_t = ku_{xx}$ with the initial value $u(x, 0) = \delta(x)$, where $\delta(x)$ is the Dirac delta function $\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$ such that $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$. The fundamental solution to this is given by $\Phi(x) = \frac{1}{\sqrt{4\pi kt}} e^{x^2/nkt}$ for some value of n. Find n.

A) -4 B) -2 C) 2 D) 4 E) NOTA

12) When finding the minimum or maximum of a function f subject to a condition $g = k$ for constant k, partial derivatives may be used via the method of Lagrange multipliers, solving the system of equations $\nabla f = \lambda \nabla g$, $g = k$. Find the maximum value of $3x + 4y + 5z$ subject to the condition $x^2 + y^2 + z^2 = 50$.

A) $\dfrac{480}{13}$ B) $\dfrac{500}{13}$ C) 40 D) 50 E) NOTA

13) Find the maximum value of $x+y+z$ subject to the conditions $x^2+y^2+z^2 = 80$ and $x+2y-3z = 14$.

A) $2\sqrt{35}$ B) $2\sqrt{42}$ C) $8\sqrt{3}$ D) $3\sqrt{22}$ E) NOTA

14) The function $N(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$ is a normal distribution. Because it is a continuous probability distribution for a real-valued variables, $\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-x^2}\, dx = 1$. Let the mode of $N(x)$ equal m. Find m.

A) 0 B) $\dfrac{1}{\sqrt{\pi}\sqrt[4]{e}}$ C) $\dfrac{1}{e\sqrt{\pi}}$ D) 1 E) NOTA

15) Let m be defined as in the previous problem. A random variable with the given normal distribution is selected. Find the probability that its value is equal to m.

A) 0 B) $\dfrac{1}{\sqrt{\pi}}$ C) $\dfrac{1}{e}$ D) $\dfrac{1}{\pi}$ E) NOTA

16) The Gaussian error function $\operatorname{erf}(x)$ is defined as $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$. Given that $\lim\limits_{x\to\infty} \operatorname{erf}(x) = 1$, evaluate $\int_0^{\infty} (1 - \operatorname{erf}(x))\, dx$.

A) $\dfrac{\sqrt{\pi}}{4}$ B) $\dfrac{1}{\sqrt{\pi}}$ C) $\dfrac{\sqrt{\pi}}{2}$ D) 1 E) NOTA

17) Austin's factory produces metal rods. The length of a randomly chosen rod is given by the continuous probability distribution $L = \frac{3\ell^2}{\ell_{\text{max}}^3}$, where $0 \le \ell \le \ell_{\text{max}}$ and ℓ_{max} is the maximum length of Austin's rods. Find the expected length of one of Austin's rods.

A) $\dfrac{2}{3}\ell_{\text{max}}$ B) $\dfrac{3}{4}\ell_{\text{max}}$ C) $\dfrac{4}{5}\ell_{\text{max}}$ D) ℓ_{max} E) NOTA

18) Mr. Roboto's factory cuts rods of length ℓ_{max} into identical pieces and sells them. Mr. Roboto can sell a rod of length $0 \le \ell \le \ell_{\text{max}}$ for $\$\sqrt{\ell}$. However, cutting a rod (or piece of a rod) into two pieces costs \$2. Find the number of pieces Mr. Roboto's rods should be cut into that would generate the most profit.

A) $\dfrac{16}{\ell_{\text{max}}}$ B) $\dfrac{16}{\sqrt{\ell_{\text{max}}}}$ C) 16 D) $16\sqrt{\ell_{\text{max}}}$ E) NOTA

19) Julio uniformly randomly picks a real number in the range $[0, 1]$. Cal uniformly randomly picks real numbers in the same range until he picks a number larger than Julio's. Find the expected number of numbers Cal must pick.

A) 2 B) e C) 3 D) $\dfrac{\pi^2}{6}$ E) NOTA

20) Victor and Randy are flipping an unfair coin with probability p to flip Heads. They will alternate flipping the coin until one of them flips Heads, and the first person to flip Heads wins \$10 from the other. Victor has the opportunity to pay \$$x$ to go first. Given that p is uniformly randomly distributed on the interval $[0,1]$, find the maximum value of x that is profitable for Victor, rounded to the nearest cent. You may use the approximations $\ln 2 = 0.693$, $\ln 3 = 1.099$.

A) \$3.33 B) \$3.47 C) \$3.86 D) \$4.06 E) NOTA

21) A Wiener process (also called a Brownian motion) is a real-valued continuous-time stochastic process, where future increments are independent of past values. Increments of a Wiener process are Gaussian; $W_{t+u} - W_t$ is normally distributed with mean 0 and variance u for $u \in \mathbb{R}^+$. Suppose a particular Wiener process has $W_0 = 0$ and $W_1 = 45$. Find the expected value of W_{45}.

A) 0 B) 1 C) 45 D) 2025 E) NOTA

For Questions $22-24$, you may use the following information. The geometric integral is denoted $\Pi_a^b f(x)^{dx}$ is equal to $\exp\left(\int_a^b \ln(f(x))\, dx\right)$, where $\exp(x) = e^x$.

22) Find $\displaystyle\prod_2^4 (3^x)^{dx}$.

A) 36 B) 72 C) 216 D) 729 E) NOTA

23) If $\displaystyle\prod_2^4 (\sqrt[3]{x})^{dx} = Ne^q$ for rational q, find Nq.

A) $-\frac{8}{3}$ B) $-\frac{4}{3}$ C) $\frac{4}{3}$ D) $\frac{8}{3}$ E) NOTA

24) Find $\displaystyle\prod_2^4 \left(e^{x^3}\right)^{dx}$.

A) e^{12} B) e^{56} C) e^{60} D) $e^{64} - e^8$ E) NOTA

25) Identify the graph of $x = \sin t$, $y = 2\cos(2t)$ *on its domain.*

A) Ellipse B) Hyperbola C) Line D) Parabola E) NOTA

26) The top part of Ryan's bald head is a half-sphere surface with radius 9 inches. As he learns even more movie trivia, his head increases in size but maintains its hemispherical shape. After 12 hours of watching the finest kino movies, the radius of his head is now 15 inches. Find the rate (in square inches per hour) the surface area of the top of his head will be increasing after four viewings of 12 Angry Men (1.5 hours each) if the radius keeps growing at the same rate.

A) 24π B) 30π C) 36π D) 54π E) NOTA

27) The Librarian has printed out and is now reading the entirety of *Homestuck*, which is the finest kino webcomic. The Librarian started from Page 1 and has counted the number of digits used to number each page. After P pages, he has counted 2025 digits. Find P.

A) 512 B) 519 C) 711 D) 1124 E) NOTA

28) Vriska quickly fills in every box in the following puzzle with an 8 to make a true equation. Unfortunately, she smudges over all of the operators and can only remember that each one was either a plus, minus, multiplication sign, or division sign, and that the expression evaluated from left to right (as opposed to using order of operations). Given $f(8) = (((8?_1 8)?_2 8)?_3 8)?_4 8 = 960$ where each $?_i$ is an operator, find $f(3)$.

$$\boxed{8} \blacksquare \boxed{8} \blacksquare \boxed{8} \blacksquare \boxed{8} \blacksquare \boxed{8} \quad = 960$$

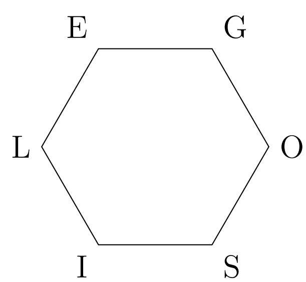
A) 24 B) 45 C) 51 D) 78 E) NOTA

29) On this test, you earn 5 points for each correct answer, 1 point for each question left blank, and 0 points for each wrong answer. This points-awarding system is in equilibrium if each of A), B), C), D), and E) is equally likely to be the answer to any problem. However, test-writers tend to prefer not making an answer E) since that means the intended answer is not shown. We believe that a student should receive confirmation that they've done good work by seeing the (hopefully correct) answer they obtain as one of the answer choices. It's no fun to work through a problem – especially a problem of considerable difficulty! – and to have to cope with the realization that the test doesn't consider your answer worthy of being given the cherished status of being one of the four answer choices. I hate to wax philosophical while writing a Calculus Applications test, but it just seems like a mean thing to do to me, so I try to avoid it. Or I'm feeling in a trolly mood and do that anyways. You never know. I've gotten less mean over time in my test-writing. You think a tilde-test is hard now? Go back to 2018 and 2019 and mock some tilde-tests. Anyways, this paragraph doesn't matter except that it talks about the scoring distribution, and instead serves as a satire of Applications tests that have half-page-long setups for one single question. Come on guys, what are we doing? Hopefully you started reading this paragraph from this end. :)

Suppose it is given that a MAΘ individual test has 3 questions whose answers are E), and the rest of the questions have answers that are randomly selected from the other four answer choices. If you guess B) on a question you can't do anything on (instead appreciating that `B)` is a cool emotion), your score is expected to be b points higher than if you left it blank. Find $\frac{1}{b}$.

A) 6 B) 8 C) 12 D) 24 E) NOTA

30) Let's close things off with some basic geometry! Polygon $LEGOSI$ is shown below. Given that $LE = EG = GO = OS = SI = 4$, $\angle LEG = \angle SIL = 45°$, $\angle EGO = \angle OSI = 180°$, and $\angle GOS = 90°$. Find the area of polygon $LEGOSI$.



(not drawn to scale)

A) 24 B) $24\sqrt{3}$ C) $32\sqrt{3}$ D) 56 E) NOTA