Answer Key

0. $\frac{4}{27}$

1. -5

2. 2

3. -16

4. $-\frac{4}{5}$

5. $4\sqrt{3}-\pi$

6. $\frac{19}{6}$

7. $\frac{93}{5}\pi$

8. $\frac{8\sqrt{3}}{27}$

9. $\frac{\pi\ln(3)}{12}$

10. 7

0. $\frac{4}{27}$ — Expand to get $f(x) = x - 2x^2 + x^3$. Taking the derivative gives $f'(x) = 1 - 4x + 3x^2 = (1 - 3x)(1 - x)$, so we have critical points at $x = \frac{1}{3}$ and $x = 1$. Checking $f(\frac{1}{3}) = \frac{4}{27}$, $f(1) = 0$, and $f(0) = 0$, giving a maximum value of $\boxed{\frac{4}{27}}$.

1. -5 — This is asking for Legosi's average vertical velocity, which is total displacement divided by total time, which is $\frac{0-20}{4} = \boxed{-5}$.

2. 2 — We first claim that $L = \lim_{x\to 0^+} x^{2x} = 1$, so the overall limit takes the indeterminate form $\frac{0}{0}$. To see this, note that $\ln(L) = \lim_{x\to 0^+} 2x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{2/x}$, which takes the indeterminate form $\frac{\infty}{\infty}$. We invoke L'Hôpital's rule to get $\lim_{x\to 0^+} \frac{1/x}{-2/x^2} = \lim_{x\to 0^+} -\frac{x}{2} = 0$, so $L = 1$, as desired.

We now invoke L'Hôpital's rule on the overall limit to get $\lim_{x\to 0^+} \frac{[x^{2x}]'}{1+\ln(x)}$. We can write the numerator as $[e^{2x\ln(x)}]' = e^{2x\ln(x)}(2\ln(x) + 2x(\frac{1}{x}) = 2x^x(1 + \ln(x))$, so the limit is simply equal to $\lim_{x\to 0^+} 2x^{2x} = \boxed{2}$.

3. -16 — The slope of the tangent line to $y = 2\sqrt{x}$ at the point $(a, \sqrt{a})$ is $\frac{1}{\sqrt{a}}$, yielding an equation $x - \sqrt{a}y = -a$, while the slope of the tangent line to $y = 3\sqrt[3]{x}$ at the point $(b, 3\sqrt[3]{b})$ is $\frac{1}{b^{2/3}}$, yielding an equation $x - b^{2/3}y = -2b$.

We want these to be the same line, and matching coefficients gives $-\sqrt{a} = -b^{2/3}$ and $-a = -2b$. Raising both sides of the first to the sixth power gives $a^3 = b^4$, and substituting $a = 2b$ gives $8b^3 = b^4$, so $b = 0$ or $b = 8$. Only $b = 8$ yields a line with positive slope, which is $x - 4y = -16$, which has x-intercept $\boxed{-16}$.

4. $-\frac{4}{5}$ — Implicit differentiation yields $2x + y + xy' + 2yy' = 0$, and solving for y' gives $y' = -\frac{2x+y}{x+2y}$. Plugging in $x = 1$ and $y = 2$ gives $\boxed{-\frac{4}{5}}$

5. $4\sqrt{3} - \pi$ — Factoring out 3 and multiplying by $\sqrt{x}$ in the numerator and denominator gives $3\int_{1/3}^{3} \frac{\sqrt{x}}{x+1}\,\mathrm{d}x$. Letting $x = u^2$, $\mathrm{d}x = 2u\,\mathrm{d}u$ gives

$$3\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{u}{u^2+1}(2u)\,\mathrm{d}u = 6\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{u^2}{u^2+1}\,\mathrm{d}u = 6\int_{1/\sqrt{3}}^{\sqrt{3}} \left(1 - \frac{1}{u^2+1}\right)\,\mathrm{d}u$$
$$= 6\left[u - \arctan(u)\right]_{1/\sqrt{3}}^{\sqrt{3}} = \boxed{4\sqrt{3} - \pi}$$

6. $\frac{19}{6}$ — Fix a coordinate system with $A = (0,0,0)$, $B = (x,0,0)$, $D = (0,y,0)$, and $A' = (0,0,z)$, where $x = 1, y = 1, z = 2, x' = 1, y' = 2, z' = 1$. The area of triangle $BA'D$ is given by $K = \frac{1}{2}\|\vec{A'B} \times \vec{A'D}\|$, where $A'B = \langle x, 0, -z\rangle$ and $A'D = \langle 0, y, -z\rangle$. Computing the cross product gives

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & 0 & -z \\ 0 & y & -z \end{vmatrix} = \langle -yz, xz, xy\rangle,$$

and thus $K = \frac{1}{2}\sqrt{y^2z^2 + x^2z^2 + x^2y^2}$. Plugging in our values of x, y, and z gives $K = \frac{3}{2}$.

To compute the derivative, it's easier to look at the square of the area:

$$K^2 = \tfrac{1}{4}(y^2z^2 + x^2z^2 + x^2y^2) \to 2KK' = \tfrac{1}{4}\sum_{\text{sym}}(2xx'y^2) \to K' = \tfrac{1}{6}\sum_{\text{sym}}(xx'y^2)$$

(where $\sum_{\text{sym}}$ consists of all combinations of x, y, and z). Plugging in our values gives the inner sum is $1 + 4 + 2 + 2 + 2 + 8 = 19$, so the rate of change of the area is $\boxed{\frac{19}{6}}$.

7. $\frac{93}{5}\pi$ — Using the disk method, the volume is given by $\pi \int_1^8 (\sqrt[3]{x})^2 \, \mathrm{d}x = \pi[\frac{3}{5}\pi x^{5/3}]_1^8 = \boxed{\frac{93}{5}\pi}$.

8. $\frac{16\sqrt{3}}{27}$ — Integrating both sides of the given identity gives that for all $1 < x < 3$,

$$\sum_{n=1}^{\infty} \left(\frac{(x-1)^n}{n \cdot 2^{n-1}} \right) = \frac{2}{3} x^{3/2}.$$

The desired sum can be obtained by plugging in $x = \frac{4}{3}$ and dividing by 2, which yields $\frac{1}{2} \cdot \frac{2}{3} \cdot (\frac{4}{3})^{3/2} = \boxed{\frac{8\sqrt{3}}{27}}$.

9. $\frac{\pi \ln(3)}{12}$ — The indefinite integral seems rough, so we'll try a bounds trick. Let $\frac{I}{\sqrt{3}} = \int_1^3 \frac{\ln(x)}{x^2+3} \, \mathrm{d}x$, and letting $x = \frac{3}{u}$, $\mathrm{d}x = -\frac{3}{u^2}\mathrm{d}u$ gives

$$\frac{I}{\sqrt{3}} = \int_3^1 \frac{\ln(\frac{3}{u})}{(\frac{3}{u})^2 + 3}(-\frac{3}{u^2})\mathrm{d}u = \int_1^3 \frac{\ln(3) - \ln(u)}{3 + u^2}\mathrm{d}u = \int_1^3 \frac{\ln(3)}{u^2+3}\mathrm{d}u - \frac{I}{\sqrt{3}},$$

so rearranging and changing variable names gives

$$I = \frac{\sqrt{3}\ln(3)}{2} \int_1^3 \frac{1}{x^2+3} \, \mathrm{d}x = \frac{\sqrt{3}\ln(3)}{2} \left[\frac{1}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}} \right) \right]_1^3 = \boxed{\frac{\pi \ln(3)}{12}}.$$

10. 7 — Letting $y = f(x)$ for simplicity, we have $xyy' = y^2 + 1$. Separating variables gives $\frac{yy'}{y^2+1} = \frac{1}{x}$. Integrating the RHS gives $\ln(x) + C$, while letting $u = y^2 + 1$ on the LHS gives $\int \frac{(1/2)du}{u} = \frac{1}{2}\ln(u) = \frac{1}{2}\ln(y^2+1)$. Raising e to both sides gives $\sqrt{y^2+1} = C'x$, which we can rearrange to $y = \pm\sqrt{C''x^2 - 1}$. Plugging in $f(1) = 1$ gives $y = \sqrt{2x^2-1}$, and plugging in $x = 5$ gives $\boxed{7}$.