1 Answer Key

1. (B)	**6.** (C)	**11.** (D)	**16.** (B)	**21.** (C)	**26.** (C)
2. (C)	**7.** (B)	**12.** (C)	**17.** (B)	**22.** (B)	**27.** (A)
3. (D)	**8.** (A)	**13.** (B)	**18.** (D)	**23.** (B)	**28.** (B)
4. (C)	**9.** (A)	**14.** (C)	**19.** (C)	**24.** (D)	**29.** (B)
5. (C)	**10.** (C)	**15.** (C)	**20.** (B)	**25.** (C)	**30.** (A)

2 Solutions

1. There are $5!/2! = 60$ total permutations of the multiset $\{F, A, A, M, T\}$. If the two vowels form an "AA" block we permute four objects $\{\text{AA}, F, M, T\}$, yielding $4! = 24$ forbidden orderings. Hence the required count is $60 - 24 = \boxed{\textbf{(B) } 36}$

2. Observe that Sharvaa's age must divide 20. However, we also know it must have been more than 6 for the second statement to be true. Therefore, Sharvaa is either 10 or 20. Since 20 does not work, we know that Sharvaa is 10 and Saathvik is 30. In $\boxed{\textbf{(C) } 10}$ years, the statement is true again.

3. Note that $\tan(\arctan 1 + \arctan 2) = \frac{1+2}{1-1\cdot 2} = \arctan(-3) = -\arctan(3)$. This cancels with the $\arctan(3)$ term to give $\tan(\arctan(4)) = \boxed{\textbf{(D) } 4}$

4. Expressing the solutions of x and y in polar form gives multiples of $\text{cis}(\frac{2\pi}{20})$ and $\text{cis}(\frac{2\pi}{25})$ respectively. Therefore, xy lies on the unit circle, and its argument is some combination of multiples of $\frac{5\pi}{50}$ and $\frac{4\pi}{50}$ modulo 2π. We conclude that this covers all multiples of $\frac{\pi}{50}$ on the interval from 0 to 2π, which is $\boxed{\textbf{(C) } 100}$.

5. Convert the cosine term:

$$4\sin^3 x - 2\sin^2 x - \sin x + \cos 2x = 4\sin^3 x - 2\sin^2 x - \sin x + 1 - 2\sin^2 x = (\sin x - 1)(2\sin x - 1)(2\sin x + 1).$$

Thus $\sin x \in \{1, \frac{1}{2}, -\frac{1}{2}\}$. On $[0, 2\pi]$ this gives

$$x = \frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6},$$

so there are $\boxed{\textbf{(C) } 5}$ solutions. It may help to note that $\sin x = 1$ is a solution, and dividing it out. This simplifies the equation to a difference of squares.

6. The chords intersect if the sectors they span overlap. Note that these sectors measure 60° and 90°, since they form an equilateral triangle and right triangle respectively with their chords. By fixing the right angle, and seeing where the other angle can go, we note there are 60 degrees of overlap at the start of the right angle and 60 degrees at the end. This adds to $\frac{120}{360} = \boxed{\textbf{(C) } 1/3}$.

7. Place the unit square so that $A(0,0),\ B(1,0),\ C(1,1),\ D(0,1),\ M(1,\frac{1}{2}),\ N(\frac{1}{2},1)$.

$$\begin{aligned} \triangle AMD: &\quad y = \tfrac{1}{2}x,\ y = -\tfrac{1}{2}x + 1; \\ \triangle ANB: &\quad y = 2x,\ y = -2x + 2,\ y = 0. \end{aligned}$$

We compute the intersections of these lines:

$$\begin{aligned} 2x = -\tfrac{1}{2}x + 1 &\quad \Longrightarrow x = \tfrac{2}{5}, \\ -\tfrac{1}{2}x + 1 = -2x + 2 &\quad \Longrightarrow x = \tfrac{2}{3}, \\ \tfrac{1}{2}x = -2x + 2 &\quad \Longrightarrow x = \tfrac{4}{5}. \end{aligned}$$

Therefore, the intersection of the triangles is a quadrilateral with vertices: $(0,0), (\frac{2}{5}, \frac{4}{5}), (\frac{2}{3}, \frac{2}{3}), (\frac{4}{5}, \frac{2}{5})$. By Shoelace Theorem, the area of this quadrilateral is $\boxed{\textbf{(B) } 4/15}$.

8. Write

$$\sum_{i=1}^{n} \frac{1}{\sqrt{n(n+i)}} = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\sqrt{1+i/n}} \xrightarrow[n\to\infty]{} \int_0^1 \frac{dt}{\sqrt{1+t}} = \Big[-2\sqrt{1+t}\Big]_0^1 = \boxed{\textbf{(A) } 2\sqrt{2} - 2}.$$

9.

$$\frac{x+y}{\frac{x}{y}+\frac{y}{x}-2}\left(3y-3x+\frac{x^3-y^3}{xy}\right)=\frac{x+y}{\frac{x^2+y^2-2xy}{xy}}\left(3(y-x)+\frac{(x-y)(x^2+xy+y^2)}{xy}\right)$$

$$=\frac{xy(x+y)}{(x-y)^2}\left((x-y)\left[-3+\frac{x^2+xy+y^2}{xy}\right]\right)$$

$$=\frac{xy(x+y)}{(x-y)^2}\cdot(x-y)\cdot\frac{(x-y)^2}{xy}$$

$$=(x+y)(x-y)=x^2-y^2.$$

With x, y distinct positive integers, $x^2-y^2=(x+y)(x-y)$ is the product of two integers of the same parity. Among the choices only $14=2\cdot 7$ mixes parity, hence $\boxed{\textbf{(A) } 14}$ cannot occur.

10. Let the roots be $a-d,\ a,\ a+d$. From Vieta,

$$3a=9, \qquad (a-d)a(a+d)=-165.$$

Thus $a=3$ and $27-3d^2=-165 \Rightarrow d^2=64$. Finally, using sum of pair of roots,

$$t=3a^2-d^2=27-64=\boxed{\textbf{(C) } -37}.$$

11. $f(x)=\frac{x^2-8}{x+3}=x-3+\frac{1}{x+3}$. Differentiate: $f'(x)=1-\frac{1}{(x+3)^2}$. Thus the extrema occur at $x=-4,-2$.

$$f(-4)=-8, \qquad f(-2)=-4.$$

Therefore

$$\frac{3s-2q}{r-p}=\frac{-24-(-8)}{-4-(-2)}=\boxed{\textbf{(D) } 8}.$$

12. Less than two impostors could have been in the last 4 people, so at least 1 impostor is in the first 6 to be thrown out. Similarly, at least 2 impostors are in the first 8 to be thrown out. If the two impostors are in the first 6, there are 15 ways. Otherwise, there are $6\cdot 2$ ways. The answer is $27/\binom{10}{2}=\boxed{\textbf{(C) } 3/5}$.

13. By the British-Flag Theorem, $AP^2+CP^2=BP^2+DP^2$ gives $1^2+3^2=5+DP^2$, so $DP=\sqrt{5}$. Place $A(0,0)$, $B(s,0)$, $C(s,s)$. Solving $x^2+y^2=1$ and $(x-s)^2+y^2=5$ yields $x=y=1/\sqrt{2}$ and $s^2-\sqrt{2}s-4=0$. Taking the positive root gives $s=2\sqrt{2}$, hence the square's area is $s^2=\boxed{\textbf{(B) } 8}$.

14. We will solve for a general parabola: $\frac{x^2}{a^2}-\frac{y^2}{b^2}$. Let AB have a slope of m. This means B lies on the line $y=mx$ and satisfies

$$\frac{x^2}{a^2}-\frac{m^2x^2}{b^2}=1 \implies x^2=\frac{a^2b^2}{b^2-a^2m^2}.$$

Hence the squared length

$$r^2=AB^2=x^2+m^2x^2=(1+m^2)\frac{a^2b^2}{b^2-a^2m^2}. \tag{1}$$

Take C on the line $y=-\frac{1}{m}x$. Repeating the same step with slope $-\frac{1}{m}$ gives

$$s^2=AC^2=\left(1+\tfrac{1}{m^2}\right)\frac{a^2b^2m^2}{b^2m^2-a^2}. \tag{2}$$

Adding the reciprocals of (1) and (2):

$$\frac{1}{r^2}+\frac{1}{s^2}=\frac{b^2-a^2m^2}{(1+m^2)a^2b^2}+\frac{b^2m^2-a^2}{(1+m^2)a^2b^2}=\frac{b^2-a^2}{a^2b^2},$$

a constant independent of m. Plugging in $a^2=9,\ b^2=36$ gives:

$$\frac{1}{r^2}+\frac{1}{s^2}=\frac{36-9}{9\cdot 36}=\frac{27}{324}=\boxed{\textbf{(C) } \frac{1}{12}}.$$

15. The given number is $1111^2 - 3^2 = 1108 \cdot 1114 = 2^3 \cdot 277 \cdot 557 \implies \boxed{\textbf{(C) } 836}$

16. The medial triangle has area $1/4$.

- Probability $1/4$: the random point lands inside it, and the hull is exactly that triangle (area $1/4$).
- Probability $3/4$: the point is in a corner sub-triangle of area $1/4$. The hull then adds the triangle with two midpoints; its expected height is one-third of the small triangle's height, so its expected area is $\frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$. The conditional hull area is $1/4 + 1/12 = 1/3$.

Hence

$$E = \frac{1}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{16} + \frac{1}{4} = \boxed{\textbf{(B) } \tfrac{5}{16}}.$$

17. Let speeds be a (Alice) and b (Bob), bridge length L.

First meeting: $\dfrac{1000}{a} = \dfrac{L - 1000}{b} \Rightarrow a/b = 1000/(L - 1000)$.

Between the two meetings Alice covers $(L - 1000) + 400 = L - 600$ and Bob $1000 + (L - 400) = L + 600$:

$$\frac{L - 600}{a} = \frac{L + 600}{b} \implies (L - 600)(L - 1000) = 1000(L + 600).$$

Solving gives $L = \boxed{\textbf{(B) } 2600}$.

18. The power series is

$$3(1(4(1(5(x + O(x^3)) + O(x^3)) + O(x^3)) + O(x^3)) + O(x^3)) = \boxed{\textbf{(D) } 60}\, x + O(x^3)$$

from which we can read out the first derivative using the coefficient of x.

19. Let the "radius" of an octahedron be the distance from the center to its vertex. An octahedron of radius r is a union of eight right-angled pyramids with three legs of length r, so its overall volume is $8r^3/6$. In our case the radius is 1 so the answer is $\boxed{\textbf{(C) } 4/3}$.

20. It is clear that S_3 is a cube, so the sequence has odd indexed terms that are cubes and even indexed terms that are octahedrons. So, we just need to find the ratio of similarity between the side lengths of S_5, S_3 which clearly matches the ratio for S_3, S_1 which can be computed to be $\frac{1}{3}$. The answer is then $1/9$, giving $\boxed{\textbf{(B) } 10}$

21. We aim to compute $|S_1| - |S_2| + |S_3| - |S_4| - \cdots$ but using our similarity argument from the previous problem this is

$$\begin{aligned}(|S_1| - |S_2|) + \frac{1}{27}(|S_1| - |S_2|) + \frac{1}{27^2}(|S_1| - |S_2|) + \cdots \\ = \frac{27}{26}(|S_1| - |S_2|) = \frac{27}{26}(8 - \frac{4}{3}) = \frac{90}{13} \implies \boxed{\textbf{(C) } 103}\end{aligned}$$

22. Since $99999^3 = (10^5 - 1)^3 = 10^{15} - 3 \cdot 10^{10} + 3 \cdot 10^5 - 1$, it has mostly 9s from digits 14 through 10, mostly 0s from digits 9 through 5, and mostly 9s from digits 4 through 0. So the average is around $\boxed{\textbf{(B) } 6}$ (it turns out to be exact.)

23.

$$\mathbb{E}\left[\int_0^x t^y \, dt\right] = \int_0^1 \int_0^1 \frac{x^{y+1}}{y+1} \, dx \, dy = \int_0^1 \frac{1}{(y+1)(y+2)} \, dy = \Big[\ln(y+1) - \ln(y+2)\Big]_0^1 = \boxed{\textbf{(B) } \ln\left(\frac{4}{3}\right)}.$$

24. Each of the intervals $(-1, -\frac{99}{101}), \ldots, (\frac{99}{101}, 1)$ admit two solutions, except for $(-\frac{1}{101}, \frac{1}{101})$ which only admits one solution at $x = 0$. Adding in $x = \pm 1$ we have 101 intervals and an answer of $\boxed{\textbf{(D) } 203}$

25. The best choice for the first pick is not hard to see: simply take the largest prime number in the pot. We can prove that this is always the best pick as follows. If the prime is not chosen, it cannot be picked in future turns since the number 1 is already gone and no other divisors are left. Also, the largest prime doesn't affect non-prime sequences since it doesn't appear as a divisor of any numbers. Therefore, the best number to pick on the first move is $\boxed{\textbf{(C) } 89}$

26. By the previous problem's logic 7 should be our first move. Then, the remaining numbers we could choose are $4, 6, 8, 9, 10$, with the primes discarded because we cannot select them again. Trying different possible sequences, we find that the strategy of selecting $9, 6, 8, 10$ in that order gets the best possible score of $\boxed{\textbf{(C)}\ 40}$.

27. $x(t)$ has slope k until it hits value $k+1$ for the first time. As such, the time for $x(t) = k$ is simply $\frac{1}{1} + \cdots + \frac{1}{k-1} \sim \ln k$. Thus the answer is on the order $\ln 10^9 \sim \boxed{\textbf{(A)}\ 20}$.

28. The prime condition enforces the relative ordering of primes to be $2, 3, 5, 7$. Therefore, we must place $1, 4, 6, 8$. We will do this sequentially. 1 has 1 possible location (in front of 2). Once 1 has been placed, the number 4 has 4 slots (before 1, before 2, before 3, or before 5). Once 4 has been placed, there are 6 slots for 6. Once 6 has been placed, there are 8 slots for 8. Therefore, our answer is $1 \cdot 4 \cdot 6 \cdot 8 = \boxed{\textbf{(B)}\ 192}$

29. Partition $[0, 210)$ into ten intervals of length 21. On the k-th interval ($k = 0, \ldots, 9$)

$$f_{21}(x) = 21k, \qquad f_{10}(21k) = 20k,$$

a constant. Hence

$$\int_0^{210} f_{10}(f_{21}(x))\, dx = \sum_{k=0}^{9} (20k)\, 21 = 420 \sum_{k=0}^{9} k = 420 \cdot 45 = \boxed{\textbf{(B)}\ 18\,900}.$$

30. Let $I = \int_0^1 \left(x - \sqrt{1-x^2}\right)^{2024} dx$ and substitute $u = \sqrt{1-x^2}$ so that $du = -\frac{x}{\sqrt{1-x^2}}\, dx$. We get

$$I = \int_0^1 (u - \sqrt{1-u^2})^{2024} \cdot \frac{u}{\sqrt{1-u^2}}\, du.$$

Now observe that

$$2I = \int_0^1 \left(1 + \frac{u}{\sqrt{1-u^2}}\right) (u - \sqrt{1-u^2})^{2024}\, du = \int_{-1}^1 a^{2024}\, da = \frac{2}{2025} \implies I = \boxed{\textbf{(A)}\ \frac{1}{2025}}.$$