

Important Instructions for this Test:

All inverse trigonometric functions are restricted to their traditional domains and ranges, and the constant $i = \sqrt{-1}$. Good luck, have fun, and as always: "NOTA" stands for "None of These Answers is correct."

1. How many of the 26 letters in the English alphabet appear in both of the words *INTEGRAL* and *DERIVATIVE*?

- A: 2 B: 3 C: 4 D: 5 E: NOTA

2. What is the area of the finite region bound by the curves $y = x^3$ and $y = x^4$?

- A: $\frac{1}{12}$ B: $\frac{1}{16}$ C: $\frac{1}{20}$ D: $\frac{1}{24}$ E: NOTA

3. Compute the derivative of $(1 + x + x^2)^2$ with respect to x , evaluated at $x = 1$.

- A: 18 B: 24 C: 30 D: 36 E: NOTA

4. Compute the following integral:

$$\int_0^1 (1 + x + x^2)^2 dx.$$

- A: 3.4 B: 3.5 C: 3.6 D: 3.7 E: NOTA

5. A square with area 18 is rotated about one of its diagonals to create a solid. What is the volume of this solid?

- A: 18π B: 24π C: 27π D: 36π E: NOTA

6. A square with area 18 is rotated about one of its diagonals to create a solid. What is the surface area of this solid?

- A: $18\pi\sqrt{2}$ B: $24\pi\sqrt{2}$ C: $27\pi\sqrt{2}$ D: $36\pi\sqrt{2}$ E: NOTA

7. The function $f(x) = |x| + \frac{1}{|x+1|}$ has two local minima at the points (a, b) and (c, d) . Compute $abcd$.

- A: -2 B: -1 C: 0 D: 1 E: NOTA

8. Which of the following is closest to 0.99^{100} ?

- A: 0.2 B: 0.4 C: 0.6 D: 0.8 E: NOTA

9. Given that r is a nonzero real number, and the series $1^r + 2^r + 3^r + \cdots$ converges, does the series $1^{1/r} + 2^{1/r} + 3^{1/r} + \cdots$ converge or diverge?

- A: Converge B: Diverge C: Either D: Neither E: NOTA

10. Compute the following integral:

$$\int_{-\infty}^{\infty} |x| - |x-1| - |x-2| + |x-3| dx.$$

- A: 3 B: 4 C: 5 D: 6 E: NOTA

11. Let $f(x) = \arctan x$ and $a = 0.01$. Which of the following numbers is the largest?

- A: $f(f(a))$ B: $f(a)$ C: $1 - f'(a)$ D: $-f''(a)$ E: NOTA

12. Given that $\int_0^1 x^a dx = a$, which of the following is closest to the value of a ?

- A: 0.6 B: 0.7 C: 0.8 D: 0.9 E: NOTA

13. If $f(x) = x^2|x|$, then compute the value of $f'''(0)$.

- A: 1 B: 2 C: 3 D: 6 E: NOTA

14. For each positive integer n , let $f(n)$ denote the remainder when n^n is divided by 3. Compute the following limit:

$$\lim_{n \rightarrow \infty} \frac{f(1) + f(2) + \cdots + f(n)}{n}.$$

- A: $\frac{1}{3}$ B: $\frac{1}{2}$ C: $\frac{2}{3}$ D: $\frac{5}{6}$ E: NOTA

15. The line $y = ax + b$ intersects the graph $y = x + \sin x$ infinitely many times. Which of the following cannot be $a + b$?

- A: -1 B: 0 C: 1 D: 2 E: NOTA

16. The global minimum and maximum of the function $f(x) = \arcsin x \cdot \arccos x$ are m and M , respectively. Compute m/M .

- A: -8 B: -4 C: -2 D: -1 E: NOTA

17. Numerically, the area of a rectangle with integer side lengths is 2 greater than its perimeter. Find the sum of all possible values for the length of the shorter side of the rectangle.

- A: 3 B: 4 C: 6 D: 7 E: NOTA

18. For all $x > 0$, consider the function

$$f(x) = x + \frac{x}{x + \frac{x}{x + \cdots}} - \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}.$$

Compute the sum of the coordinates of the unique inflection point of $y = f(x)$.

- A: 1 B: 2 C: 3 D: 4 E: NOTA

19. The circle $x^2 + y^2 = 1$ and the parabola $y = x^2$ intersect at two points. The two lines tangent to the parabola at these two points intersect at $(0, a)$. Which of the following is closest to a ?

- A: -0.5 B: -0.6 C: -0.7 D: -0.8 E: NOTA

20. The polar graphs $r = \arccos \theta$ and $r = \arcsin \theta$ intersect at two points. Which of the following is closest to the distance between these two points?

- A: 0.7 B: 0.8 C: 0.9 D: 1.0 E: NOTA

21. The polynomial $x^5 + ax^4 + bx^3 + x^2 + 1$ is divisible by $(x + 1)^2$. Compute ab .

- A: 42 B: 56 C: 64 D: 84 E: NOTA

22. Let (A, B) be the unique point on the curve $(x^2 + y^2)^2 = 2025x$ such that B is maximized. Which of the following is closest to B/A ?

- A: 1.5 B: 1.6 C: 1.7 D: 1.8 E: NOTA

23. Sharvaa is pouring water into his empty cylindrical water bottle with radius 2 and height 10. The water flows into the bottle at a rate of π cubic units per second. However, unbeknownst to him, Albert poked a peculiar hole into the water bottle. If the water bottle contains W units of water, then water flows out the hole at a rate of $W/40$ cubic units per second. How many seconds will it take for Sharvaa to fill his bottle?

- A: $\ln 10$ B: $\ln 20$ C: $\ln 30$ D: $\ln 40$ E: NOTA

24. For each positive integer x , let $\pi(x)$ denote the number of prime numbers less than or equal to x . The Prime Number Theorem asserts that $\lim_{x \rightarrow \infty} \frac{\pi(x)}{x/\ln x} = 1$. Using this information, which of the following is closest to $\pi(10^{2025})$?

- A: 10^{2020} B: $10^{2020.5}$ C: 10^{2021} D: $10^{2021.5}$ E: NOTA

25. Define $\pi(x)$ as in Question 24. Compute the following limit:

$$\lim_{x \rightarrow \infty} \frac{\pi(x) \cdot \pi(x^3)}{\pi(x^2)^2}.$$

- A: $4/3$ B: $3/2$ C: 2 D: 3 E: NOTA

26. Compute the following limit:

$$\lim_{x \rightarrow 0} \left(\frac{1}{\sin^2 x} - \frac{\cos x}{x^2} \right).$$

- A: $1/2$ B: $3/4$ C: $5/6$ D: $7/8$ E: NOTA

27. Let $f(x) = \ln(1 + x + x^2 + x^3 + x^4 + x^5) - \ln(1 + x + x^2 + x^3 + x^4)$. Which of the following terms in the Maclaurin series of $f(x)$ has a nonzero coefficient?

- A: x^{101} B: x^{102} C: x^{103} D: x^{104} E: NOTA

28. Compute the value of the integral

$$\int_1^{\infty} \frac{1 + \ln x}{1 + x^x} dx.$$

- A: $\ln 2$ B: 1 C: $\ln 3$ D: 2 E: NOTA

29. For each positive integer n , let $f(n)$ denote the number of solutions to the equation $x + y + z = n$, where x, y , and z are nonnegative integers. There exist unique positive rational numbers r and s such that $\lim_{n \rightarrow \infty} \frac{f(n)}{rn^s} = 1$. Compute $r + s$.

- A: $7/3$ B: $5/2$ C: $10/3$ D: $7/2$ E: NOTA

30. Let R be the region in the first quadrant bounded by the coordinate axes and the curve $e^x + e^y = e^{x+y}$. Which of the following is the area of R ?

- A: $\frac{\pi^2}{12}$ B: $\frac{\pi^2}{9}$ C: $\frac{\pi^2}{8}$ D: $\frac{\pi^2}{6}$ E: NOTA