Answer Key:

1. D
2. C
3. A
4. D
5. A

6. A
7. C
8. B
9. B
10. B

11. D
12. A
13. E
14. D
15. A

16. A
17. D
18. A
19. B
20. B

21. A
22. C
23. E
24. D
25. A

26. C
27. B
28. A
29. B
30. D

Solutions:

1. D: The letters A, E, I, R, T appear in both words.

2. C: Note that the two curves intersect at $x = 0$ and $x = 1$, so the area of the region is given by

$$\int_0^1 x^3 - x^4\,dx = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}.$$

3. A: Using Chain Rule gives the derivative as $2(1 + x + x^2)(1 + 2x)$, and substituting $x = 1$ gives $2 \cdot 3 \cdot 3 = 18$.

4. D: Expanding and using the Power Rule gives

$$\int_0^1 (1 + x + x^2)^2\,dx = \int_0^1 1 + 2x + 3x^2 + 2x^3 + x^4\,dx = 1 + \frac{2}{2} + \frac{3}{3} + \frac{2}{4} + \frac{1}{5} = 3 + 0.5 + 0.2 = 3.7.$$

5. A: Note that rotating a square about one of its diagonals gives two cones connected at their bases. Since the square has area 18, its side length is $3\sqrt{2}$, so half of its diagonal length is 3. This is also the base radius and height of both cones, so the desired volume is $2 \cdot \frac{1}{3}\pi \cdot 3^2 \cdot 3 = 18\pi$.

6. A: Continuing from the previous solution, we now need to compute the lateral surface area of these cones. Their slant height is $3\sqrt{2}$ and their radius is 3, so the combined surface area is $2 \cdot \pi \cdot 3\sqrt{2} \cdot 3 = 18\pi\sqrt{2}$.

7. C: Consider the left and right-hand side derivatives at $x = 0$. For $x > 0$, $f(x) = x + \frac{1}{x+1} \Rightarrow f'(x) = 1 - \frac{1}{(x+1)^2} \Rightarrow \lim_{x\to 0^+} f'(x) = 0$. For $x < 0$, $f(x) = -x + \frac{1}{x+1} \Rightarrow f'(x) = -1 - \frac{1}{(x+1)^2} \Rightarrow \lim_{x\to 0^-} f'(x) = -2 < 0$. In particular, both of these values means that there is a local minimum at $x = 0$, so the final product is also 0.

8. B: Note that $0.99^{100} = \left(1 - \frac{1}{100}\right)^{100}$ is quite close to $\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^n = \frac{1}{e} \approx \frac{1}{2.718} \approx 0.4$.

9. B: Note that $1^r + 2^r + 3^r + \cdots$ only converges if and only if $r < -1$ by the p-series test. In particular, since $r < -1$ from the given info, we must have $\frac{1}{r} > -1$, so the second series must diverge.

10. B: Note that we can turn the integrand into a piecewise function by taking cases on the absolute value to get

$$|x| - |x-1| - |x-2| + |x-3| = \begin{cases} 0 \; if \; x \le 0 \\ 2x \; if \; 0 < x \le 1 \\ 2 \; if \; 1 < x \le 2 \\ 6 - 2x \; if \; 2 < x \le 3 \\ 0 \; if \; x > 3 \end{cases}.$$

Therefore, we can change the bounds of integration to 0 and 3. Graphing the region gives that we are simply trying to find the area of a trapezoid with base lengths 1 and 3 with height 2. The area is $\frac{1}{2}(1 + 3)2 = 4$.

11. D: Since $a = 0.01 \approx 0$, we can use the Maclaurin series for $f(x)$ approximations. Note that $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$. In particular, note that $f(x) < x$ for all $x > 0$. This means that $a > f(a) > f(f(a))$. Note that $1 - f'(a) \approx 1 - (1 - a^2) = a^2$ and $-f''(a) \approx -(-2a) = 2a$. Out of these, the largest is clearly $-f''(a)$.

12. A: Note that $\int_0^1 x^a\,dx = \frac{1}{a+1}$, so $\frac{1}{a+1} = a \Rightarrow a^2 + a - 1 = 0 \Rightarrow a = \frac{-1\pm\sqrt{5}}{2}$. The negative solution will not work, as the integral is not well defined. Therefore, $a = \frac{-1+\sqrt{5}}{2} \approx 0.6$.

13. E: When $x > 0$, we have $f(x) = x^3 \Rightarrow f'''(x) = 6$. When $x < 0$, we have $f(x) = -x^3 \Rightarrow f'''(x) = -6$. Therefore, $f'''(0)$ is not well defined.

14. D: Note that $f(1) = 1, f(2) = 1, f(3) = 0, f(4) = 1, f(5) = 2$, and $f(6) = 0$. We claim that f has a period of 6. To prove this, we use modular arithmetic and Fermat's Little Theorem, which tells us that $n^3 \equiv n$ modulo 3. This means that in modulo 3, we have

$$(n+6)^{n+6} \equiv n^{n+6} \equiv n^{n-1} \cdot n \cdot n^3 \cdot n^3 \equiv n^{n-1} \cdot n \cdot n \cdot n \equiv n^{n-1} \cdot n^3 \equiv n^{n-1} \cdot n \equiv n^n.$$

Therefore, letting $n = 6a + b$ (and letting $a \to \infty$) gives

$$\frac{f(1) + f(2) + \cdots + f(n)}{n} = \frac{a\big(f(1) + f(2) + f(3) + f(4) + f(5) + f(6)\big) + f(1) + \cdots + f(b)}{6a + b}$$
$$= \frac{5a + f(1) + \cdots + f(b)}{6a + b}.$$

Note that as $n \to \infty$, $a \to \infty$ and b always lies in $[0,5]$, so the desired limit is $\frac{5a}{6a} = \frac{5}{6}$.

15. A: Note that the lines $y = x - 1$, $y = x$, and $y = x + 1$ intersect the given graph infinitely many times, ruling out answer choices B, C, and D, respectively. We claim that the slope of all such lines must be $a = 1$. In particular, if $a \neq 1$, then $x + \sin x = ax + b \Rightarrow \sin x = (a - 1)x + b$, where $a - 1 \neq 0$. In particular, since $\sin x \in [-1,1]$, then we must have $(a - 1)x + b \in [-1,1]$, bounding x in a finite interval. Since the graph $y = x + \sin x$ does not have infinite oscillatory behavior, if must intersect the line in the finite interval only a finite number of times, contradiction. Since $a = 1$, in order for $x + \sin x = ax + b = x + b$, we must have $b \in [-1,1]$. In particular, $a + b \in [0,2]$, and -1 is not in this interval.

16. A: Note that $f'(x) = \frac{\arccos x - \arcsin x}{\sqrt{1-x^2}}$, which is only 0 when $\arccos x = \arcsin x$, which only happens at $x = \sqrt{2}/2$ and $f(x) = \pi^2/16$. We also need to check the boundary of the domain, namely $f(-1) = -\pi^2/2$ and $f(1) = 0$. This means that $m = -\pi^2/2$ and $M = \pi^2/16$, so the answer is $\frac{m}{M} = -8$.

17. D: Let the side lengths of the rectangle be a and b with $a \leq b$. Then $ab = 2(a + b) + 2$ and $ab - 2a - 2b = 2$. Using Simon's Favorite Factoring Trick, $(a - 2)(b - 2) = 6 = 1 \cdot 6 = 2 \cdot 3$. Therefore, $a = 3$ or $a = 4$ and the sum of all possible a is 7.

18. A: We can evaluate the continued fraction and infinite radical as follows:

$$y = x + \frac{x}{x + \cdots} \Rightarrow y = x + \frac{x}{y} \Rightarrow y^2 - xy - x = 0 \Rightarrow y = \frac{x + \sqrt{x^2 + 4x}}{2},$$
$$z = \sqrt{x + \sqrt{x + \cdots}} \Rightarrow z = \sqrt{x + z} \Rightarrow z^2 - z - x = 0 \Rightarrow z = \frac{1 + \sqrt{1 + 4x}}{2}.$$

This means that taking the second derivative of $f(x) = y - z$ gives

$$f''(x) = \frac{1}{2}\left[-\frac{4}{(x^2 + 4x)^{\frac{3}{2}}} + \frac{4}{(1 + 4x)^{\frac{3}{2}}}\right].$$

The only inflection points occur when $x^2 + 4x = 1 + 4x \Rightarrow x = 1$, and since $f(1) = 0$, our answer is $1 + 0 = 1$.

19. B: The two given curves intersect at $x^2 + x^4 = 1$, which has two real solutions, $x = \pm a$, where $a^2 = \frac{-1+\sqrt{5}}{2}$. Therefore, by the symmetry of the parabola, the tangent lines of these intersection points meet on the y-axis. The derivative of $y = x^2$ at $x = a$ is $2a$, so one of the tangent lines is $y - a^2 = 2a(x - a)$. The y-intercept of this line is given by $(0, -a^2)$. Therefore, our answer is $-a^2 = \frac{1-\sqrt{5}}{2} \approx -0.618 \approx -0.6$.

20. B: Note that these two curves intersect at the origin because the polar points $\left(0, \frac{\pi}{2}\right)$ and $(0,0)$ pass through these curves, respectively. The second place of intersection occurs when $r = \arccos\theta = \arcsin\theta$, and since $\arccos\theta + \arcsin\theta = \frac{\pi}{2}$ for all $\theta \in [-1,1]$, this must happen when $r = \arccos\theta = \arcsin\theta = \frac{\pi}{4}$. Since they intersect on $r = \frac{\pi}{4}$ and the origin, the distance between these two points is $\frac{\pi}{4} \approx \frac{3.14}{4} \approx 0.8$.

21. A: By the Remainder Theorem, substituting in $x = -1$ gives that $a - b + 1 = 0$. Note that the Product Rule for Derivatives tells us that taking the derivative of the original polynomial is also a multiple of $x + 1$. Using the Remainder Theorem again tells us that $-4a + 3b + 3 = 0$. Solving these two equations gives $(a, b) = (6,7)$, and their product is 42.

22. C: Doing implicit differentiation (and setting the derivative to be 0) gives

$$2(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 2025 \Rightarrow 4x(x^2 + y^2) = 2025 \Rightarrow$$
$$4x^2(x^2 + y^2) = 2025x = (x^2 + y^2)^2.$$

It is clear that the y-coordinate is not maximized at the origin, so we can divide by $x^2 + y^2$ to get $4x^2 = x^2 + y^2$, so $y^2 = 3x^2$. Note that $x = \frac{(x^2+y^2)^2}{2025}$ cannot be negative, and at the maximum y-coordinate, $y > 0$. This means that $B = A\sqrt{3} \Rightarrow \frac{B}{A} = \sqrt{3} \approx 1.732$.

23. E: Let $y = y(t)$ denote the amount of water in Sharvaa's water bottle at time t. At time t, the water bottle is being filled at a rate of π cubic units per second and being emptied at a rate of $y/40$ cybic units per second, so we have $\frac{dy}{dt} = \pi - \frac{y}{40}$. We can use separation to solve this differential equation to get

$$\frac{1}{\pi - \frac{y}{40}}dy = dt \Rightarrow -40\ln\left(\pi - \frac{y}{40}\right) = t + C \Rightarrow y = 40\pi - 40e^{-\frac{t+C}{40}}.$$

However, regardless of the values of C or t, the exponential portion of the right-hand side is always positive meaning that $y(t) < 40\pi$, the volume of the entire water bottle, meaning that it will never be full.

24. D: Plugging in the given information gives that $\pi(10^{2025}) \approx \frac{10^{2025}}{\ln(10^{2025})} = \frac{10^{2025}}{2025\ln 10} \approx \frac{10^{2025}}{10^{3.5}} = 10^{2021.5}$.

25. A: Using the Prime Number Theorem asymptotic formula gives

$$\lim_{x\to\infty}\frac{\pi(x)\cdot\pi(x^3)}{\pi(x^2)^2} = \lim_{x\to\infty}\frac{\frac{x}{\ln x}\cdot\frac{x^3}{3\ln x}}{\left(\frac{x^2}{2\ln x}\right)^2} = \lim_{x\to\infty}\frac{x^4/(3(\ln x)^2)}{x^4/(4(\ln x)^2)} = \frac{4}{3}.$$

26. C: Using the Maclaurin series expansion gives

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x} - \frac{\cos x}{x^2}\right) = \lim_{x\to 0}\left(\frac{1}{\left(x - \frac{x^3}{6}\right)^2} - \frac{1 - \frac{x^2}{2}}{x^2}\right) = \lim_{x\to 0}\left(\frac{1}{\left(x - \frac{x^3}{6}\right)^2} - \frac{1}{x^2} + \frac{1}{2}\right) =$$

$$\frac{1}{2}+\lim_{x\to 0}\left(\frac{1-\left(1-\frac{x^2}{6}\right)^2}{\left(x-\frac{x^3}{6}\right)^2}\right)=\frac{1}{2}+\lim_{x\to 0}\left(\frac{x^2/3}{x^2}\right)=\frac{1}{2}+\frac{1}{3}=\frac{5}{6}.$$

27. B: Note that

$$f(x)=[\ln(1+x+x^2+x^3+x^4+x^5)+\ln(1-x)]-[\ln(1+x+x^2+x^3+x^4)+\ln(1-x)]$$
$$=\ln(1-x^6)-\ln(1-x^5).$$

The Maclaurin series of $\ln(1-x)$ contains all of the nonzero powers of x, meaning that the Maclaurin series of $\ln(1-x^6)$ contains all of the nonzero powers of x^6. Similarly, $\ln(1-x^5)$ contains all of the nonzero powers of x^5. Subtracting these gives all nonzero powers of x^5 and x^6, one of which is x^{102}.

28. A: Using the u-substitution $u=x^x \Rightarrow du=u(1+\ln x)dx$ gives

$$\int_1^\infty \frac{1+\ln x}{1+x^x}dx=\int_1^\infty \frac{1}{1+u}\cdot\frac{1}{u}du=\int_1^\infty \frac{1}{u}-\frac{1}{u+1}du=\ln\left(\frac{u}{u+1}\right)\Big|_1^\infty=\ln 2.$$

29. B: The number of solutions to $x+y+z=n$ is $(n+3-1)C_{3-1}=\frac{(n+1)(n+2)}{2}\approx\frac{1}{2}n^2$. We have $r+s=\frac{1}{2}+2=\frac{5}{2}$.

30. D: Solving for y gives $y=\ln\left(\frac{e^x}{e^x-1}\right)$, so the area of R is given by

$$\int_0^\infty \ln\left(\frac{e^x}{e^x-1}\right)dx=\int_1^\infty \ln\left(\frac{u}{u-1}\right)\cdot\frac{1}{u}du=\int_0^1 \ln\left(\frac{1/y}{1/y-1}\right)\cdot\frac{1}{y}dx=\int_0^1\frac{-\ln(1-y)}{y}dy.$$

Here, we do the u-substitutions $u=e^x$ and $y=1/u$. We can use the Taylor series for the final integral to get

$$\int_0^1\frac{-\ln(1-y)}{y}dy=\int_0^1\left(1+\frac{y}{2}+\frac{y^2}{3}+\cdots\right)dy=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}.$$