Answer Key:

1. B
2. A
3. D
4. C
5. C

6. B
7. A
8. D
9. C
10. A

11. B
12. D
13. C
14. B
15. A

16. D
17. C
18. A
19. D
20. D

21. C
22. B
23. C
24. A
25. D

26. C
27. B
28. A
29. A
30. B

Solutions:

1. B: $\frac{d}{dx}\left[14\pi\sin\left(\sqrt{x}\right)\right]_{x=\pi^2} = \left[14\pi\cos\left(\sqrt{x}\right)\left(\frac{1}{2\sqrt{x}}\right)\right]_{x=\pi^2} = \boxed{-7}$.

2. A: $\lim_{x\to\infty}\arctan\left(\frac{\sqrt{3}x^4}{15-3x^4}\right) = \arctan\left(\frac{-\sqrt{3}}{3}\right) = \boxed{-\frac{\pi}{6}}$.

3. D: In the Limit definition of a derivative, the Δx in the denominator must match the difference in the points evaluated in the numerator. Therefore: $\lim_{h\to 0}\frac{\tan\left(\frac{\pi}{3}+5h\right)-\tan\left(\frac{\pi}{3}-3h\right)}{2h} = 4\cdot\lim_{h\to 0}\frac{\tan\left(\frac{\pi}{3}+5h\right)-\tan\left(\frac{\pi}{3}-3h\right)}{8h} = 4\cdot \frac{d}{dx}[\tan(x)]\Big]_{x=\frac{\pi}{3}} = 4\sec^2\left(\frac{\pi}{3}\right) = \boxed{16}$.

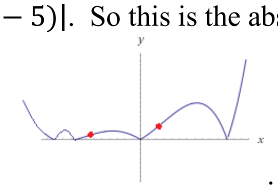
4. C: According to the Mean Value Theorem for Derivatives, $f'(1) = \frac{f(b)-f(0)}{b-0} \to -2 = \frac{1-b^2-1}{b} = -b \to b = \boxed{2}$.

5. C: Since $f(x)$ is quadratic, the sum of the solutions is $S = -\frac{b}{a}$ and the product is $P = \frac{1-ab}{a}$. Since $S = 2025$ we have $b = -2025a$. So $P = \frac{1-a(-2025a)}{a} = \frac{1}{a} + 2025a \to P' = -\frac{1}{a^2} + 2025$. To find the minimum, $P' = -\frac{1}{a^2} + 2025 = 0 \to a = \frac{1}{45}$. So $P = 45 + 45 = \boxed{90}$.

6. B: $\int_0^2 f(x)dx = A + B = 20$, $\int_1^3 f(x)dx = B + C = 25$, and $\int_0^3 f(x)dx = A + B + C = 9$. Therefore $\int_1^2 f(x)dx = B = (A+B) + (B+C) - (A+B+C) = 20 + 25 - 9 = \boxed{36}$.

7. A: The slope of the hyperbola is given by $y + xy' = 0 \to y' = -\frac{y}{x}$. The slope of the ellipse is given by $\frac{2}{a^2}x + 2yy' = 0 \to y' = -\frac{1}{a^2}\frac{x}{y}$. Therefore the simultaneous equations $\frac{x^2}{a^2} + y^2 = \frac{xy}{25}$ and $-\frac{y}{x} = -\frac{1}{a^2}\frac{x}{y}$ must have one solution. The second equations yields $x = ay$. So $\frac{a^2y^2}{a^2} + y^2 = ay\cdot\frac{y}{25} \to 2y^2 = \frac{a}{25}y^2 \to a = \boxed{50}$.

8. D: $f(x) = \frac{x}{1+\frac{x}{1+\frac{x}{1+\frac{x}{\cdots}}}} \to y = \frac{x}{1+y} \to y^2 + y - x = 0 \to (2y+1)y' = 1 \to y' = \frac{1}{2y+1}$. When $x = 20$, $y^2 + y - 20 = 0 \to (y+5)(y-4) = 0 \to y = 4$ since $f(x) > 0$ when $x > 0$. Therefore $y' = \boxed{\frac{1}{9}}$.

9. C: $f(x) = |3x^4 - 5x^3 - 15x^2 + 25x| = |x(3x^3 - 5x^2 - 15x + 25)| = |x(3x(x^2-5) - 5(x^2-5))| = |x(3x-5)(x^2-5)|$. So this is the absolute value of a positive quartic with four distinct roots, which will look something like: 

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It will have two points of inflection not at the roots (red dots). This graph has three relative maxima, four relative minima, and changes concavity six times. So the answer is $\boxed{72}$.

10. C: The Maclaurin series for $e^{x^5} = \sum_{k=0}^{\infty}\frac{x^{5k}}{k!}$. By definition, the Maclaurin series takes the form $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)x^n}{n!}$. Therefore for $n = 2025$, $\frac{f^{(2025)}(0)x^{2025}}{2025!} = \frac{x^{5k}}{k!}$. Therefore we are looking at $k = 405$ and so

$f^{(2025)}(0) = \frac{(2025!)}{(405!)} = 2025 \cdot 2024 \cdot \dots \cdot 407 \cdot 406$. To find the number of factors of five, we need to divide: $2025 \div 5 = 405\ R\ 0, 2025 \div 25 = 81\ R\ 0, 2025 \div 125 = 16\ R\ 25, 2025 \div 625 = 3\ R\ 150$. The total of the quotients is 505. However, we need to do the same thing for 405 and remove those factors: $405 \div 5 = 81\ R\ 0, 405 \div 25 = 16\ R\ 5, 405 \div 125 = 3\ R\ 30$. These quotients add up to 100, so the final answer is $\boxed{405}$.

11. B: Minimizing the distance will also minimize the distance squared, so it suffices to minimize the distance squared: $Dsq = \left(x - \frac{1}{2}\right)^2 + \left(x^{\frac{3}{2}} - 1 + 1\right)^2 \to Dsq' = 2\left(x - \frac{1}{2}\right) + 3x^2 = 3x^2 + 2x - 1 = 0 \to x = \frac{-2 \pm \sqrt{4+12}}{6} = -1\ or\ \frac{1}{3}$. $Dsq'' = 6x + 2$ so at $x = \frac{1}{3}$ Dsq'' is positive and we have a minimum. The minimum distance squared is therefore $Dsq = \left(\frac{1}{3} - \frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^3 = \frac{1}{36} + \frac{1}{27} \to D = \frac{\sqrt{36+27}}{18\sqrt{3}} = \frac{\sqrt{7}}{6\sqrt{3}} = \boxed{\frac{\sqrt{21}}{18}}$.

12. D: $\int_1^2 (12x^3 - 6x^2 + 8x - 3)dx = [3x^4 - 2x^3 + 4x^2 - 3x]_1^2 = (48 - 16 + 16 - 6) - (3 - 2 + 4 - 3) = \boxed{44}$.

13. C: $f(x) = g(x) \to 20x^3 - 25x^4 = 0 \to 5x^3(4 - 5x) = 0 \to x = 0\ or\ \frac{4}{5}$. Therefore the area is $\int_0^{4/5} 20x^3 - 25x^4 dx = [5x^4 - 5x^5]_0^{4/5} = \frac{4^4}{5^3}\left(1 - \frac{4}{5}\right) = \frac{4^4}{5^4} = \boxed{\frac{256}{625}}$.

14. B: Simpson's Rule is exact for cubics, so the answer is just $\int_1^2 4x^3 + 3x^2 + 2x + 1\, dx = [x^4 + x^3 + x^2 + x]_1^2 = 16 + 8 + 4 + 2 - 1 - 1 - 1 - 1 = \boxed{26}$.

15. A: The desired volume is $2\pi \int_0^1 x^{25}(1 - x)dx = 2\pi\left[\frac{1}{26}x^{26} - \frac{1}{27}x^{27}\right]_0^1 = \boxed{\frac{\pi}{351}}$.

16. D: $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{4n^2 - k^2}} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \frac{1}{\sqrt{4 - \left(\frac{k}{n}\right)^2}} = \int_0^1 \frac{1}{\sqrt{4 - x^2}} dx = \arcsin\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{6}}$

17. C: The volume of the water is given by $V = 625\pi(2r) - \frac{4}{3}\pi r^3 \to V' = 0 = 1250\pi - 4\pi r^2 \to r = \sqrt{\frac{1250}{4}} = \boxed{\frac{25\sqrt{2}}{2}}$

18. A: $\int_0^1 \frac{x^3}{\sqrt{x^4+1}} dx$. Let $\tan(u) = x^2$. Then $x^4 + 1 = \sec^2(u)$ and $xdx = \frac{1}{2}\sec^2(x)$. So $\int_0^1 \frac{x^3}{x^4+1} dx = \frac{1}{2}\int_0^{\pi/4} \tan(u)\sec(u)\, du = \frac{1}{2}[\sec(u)]_0^{\pi/4} = \frac{1}{2}(\sqrt{2} - 1) = \frac{-1+\sqrt{2}}{2}$. We know the roots of the polynomial are $\frac{-4 \pm \sqrt{16 - 16K}}{8} = \frac{-1 \pm \sqrt{1-K}}{2}$ so therefore $K = \boxed{-1}$.

19. D: $\int_0^1 h(x)dx = \int_0^1 h(x)\frac{F(x)+G(x)}{F(x)+G(x)}dx = \int_0^1 \frac{h(x)F(x)+h(x)G(x)}{F(x)+G(x)}dx = \int_0^1 \frac{F\prime(x)+G\prime(x)}{F(x)+G(x)}dx = \ln|F(x) + G(x)|]_0^1 = \ln\left(\frac{25}{7} + \frac{24}{7}\right) - \ln(1 + 0) = \boxed{\ln(7)}$.

20. D: Letting $u = \int_0^x f(t)dt$ we know from the chain rule that $\frac{d}{du}[f(x)] = f'(x)\frac{dx}{du} = \frac{f'(x)}{\frac{du}{dx}}$. Since $u = \int_0^x f(t)dt \to \frac{du}{dx} = f(x)$ by the fundamental theorem of calculus, $\frac{d}{d(\int_0^x f(t)dt)}[f(x)] = \frac{f'(x)}{f(x)}$. Therefore $\int_0^1 \frac{d}{d(\int_0^x f(t)dt)}[f(x)]dx = \int_0^1 \frac{f'(x)}{f(x)}dx = \ln|f(1)| - \ln|f(0)| = \ln\left(\frac{25}{20}\right) = \boxed{\ln\left(\frac{5}{4}\right)}$.

21. C: Asymptotically we need $3x+y+1-\frac{2x-y^2+2}{2x}<-1 \to 6x^2+2xy+2x-2x+y^2-2+1<0 \to 6x^2+2xy+y^2<1$. This interior of an ellipse will have area $\frac{2\pi}{\sqrt{4(6)(1)-2^2}}=\boxed{\frac{\pi}{\sqrt{5}}}$.

22. B: Note that for each differential equation, the solution is given by $y_k'+f_k(x)y_k=0 \to e^{\int_1^x f_k(t)dt}y_k'+f_k(x)e^{\int_1^x f_k(t)dt}y_k=0 \to \left[e^{\int_1^x f_k(t)dt}y_k\right]'=0 \to y_k=Ce^{-\int_1^x f_k(t)dt}$ with $y_k(1)=C$. Therefore $\frac{y_1y_2^2}{y_3}=\frac{2e^{-\int_1^x e^t\ln(t)dt}\cdot 25e^{-2\int_1^x \frac{xe^t}{t}dt}}{10e^{-\int_1^x \frac{xe^t}{t^2}dt}}=5e^{-\int_1^x e^t\ln(t)dt-\int_1^x\frac{xe^t}{t}dt-\int_1^x\frac{xe^t}{t}dt+\int_1^x\frac{xe^t}{t^2}dt}=5e^{-\left(\int_1^x e^t\left(\ln(t)+\frac{1}{t}\right)dt\right)-\left(\int_1^x e^t\left(\frac{1}{t}-\frac{1}{t^2}\right)dt\right)}=5e^{-\left[e^t\ln(t)\right]_1^x-\left[\frac{e^t}{t}\right]_1^x}=5e^{-e^x\ln(x)-\frac{e^x}{x}+e}$. Therefore $\int_1^e \ln\left(\frac{y_1y_2^2}{y_3}\right)dx=\int_1^e\left(\ln(5)-\left(e^x\ln(x)+\frac{e^x}{x}\right)+e\right)dx=(e-1)(\ln(5)+e)-\left[e^x\ln(x)\right]_1^e=\boxed{(e-1)(\ln(5)+e)-e^e}$

23. C: The eigenvalues of $\begin{bmatrix} x & x^3 \\ \frac{1}{x} & 2x\end{bmatrix}$ are given by $(x-\gamma)(2x-\gamma)-x^2=0 \to \gamma^2-3x\gamma+x^2=0 \to \gamma=\frac{3x\pm\sqrt{9x^2-4x^2}}{2}=\frac{(3\pm\sqrt{5})x}{2}$. The eigenvector associated with $\frac{(3+\sqrt{5})x}{2}$ is given by $\begin{bmatrix}\left(\frac{-1-\sqrt{5}}{2}\right)x & x^3 \\ \frac{1}{x} & \left(\frac{1-\sqrt{5}}{2}\right)x\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}=\mathbf{0} \to \boldsymbol{v_1}=\begin{bmatrix}x^2\\ \left(\frac{1+\sqrt{5}}{2}\right)\end{bmatrix}$. Similarly, the eigenvector associated with $\frac{(3-\sqrt{5})x}{2}$ is given by $\begin{bmatrix}\left(\frac{-1+\sqrt{5}}{2}\right)x & x^3 \\ \frac{1}{x} & \left(\frac{1+\sqrt{5}}{2}\right)x\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}=\mathbf{0} \to \boldsymbol{v_2}=\begin{bmatrix}x^2\\ \left(\frac{1-\sqrt{5}}{2}\right)\end{bmatrix}$. The smaller angle between these two vectors is given by $\cos(\theta)=\frac{\boldsymbol{v_1}\cdot\boldsymbol{v_2}}{|\boldsymbol{v_1}||\boldsymbol{v_2}|}=\frac{x^4-1}{\sqrt{x^4+3+\sqrt{5}}\sqrt{x^4+3-\sqrt{5}}}=\frac{x^4-1}{\sqrt{x^8+6x^4+4}} \to -\sin(\theta)\,\theta'=\frac{\left(\sqrt{x^8+6x^4+4}\right)(4x^3)-(x^4-1)\left(\frac{8x^7+24x^3}{2\sqrt{x^8+6x^4+4}}\right)}{x^8+6x^4+4}x'$. The eigenvectors are perpendicular when $\cos(\theta)=0 \to \sin(\theta)=1, x=1$, so$\to -\theta'=\frac{(\sqrt{11})(4)}{11}(\sqrt{11}) \to \theta'=\boxed{-4}$.

24. A: Note that $x^y<y^x \leftrightarrow x^{1/x}<y^{1/y}$ by raising both sides to the $\frac{1}{xy}$. Now consider the function $f(x)=x^{1/x} \to \ln(f)=\frac{\ln(x)}{x} \to \frac{f'}{f}=\frac{x\left(\frac{1}{x}\right)-\ln(x)}{x^2} \to f'(x)=x^{\frac{1}{x}-2}(1-\ln(x))=0$ when $x=e$. Further, since $f'(1)>0$ but $f'(e^2)<0$, $x=e$ is a maximum, f is decreasing for $x>e$, and therefore for all $b>a>e$, $a^{1/a}>b^{1/b}$. Now there are several ways to determine that $\sqrt{2}+\sqrt{3}>\pi$, but one is to note that $\sqrt{2}+\sqrt{3}>\frac{22}{7}>\pi$, with the second inequality being a known fact. The first is shown by noting that $\left(\sqrt{2}+\sqrt{3}\right)^2=5+2\sqrt{6} \to \left(\frac{\left(\sqrt{2}+\sqrt{3}\right)^2-5}{2}\right)^2=6$ but $\left(\frac{22}{7}\right)^2=\frac{484}{49} \to \left(\frac{\left(\frac{22}{7}\right)^2-5}{2}\right)^2=\left(\frac{239}{98}\right)^2=\frac{57121}{9604}<6$. In any case since $\sqrt{2}+\sqrt{3}>\pi \to \pi^{1/\pi}>\left(\sqrt{2}+\sqrt{3}\right)^{\frac{1}{\sqrt{2}+\sqrt{3}}} \to \boxed{\pi^{\sqrt{2}+\sqrt{3}}}>\left(\sqrt{2}+\sqrt{3}\right)^{\pi}$

25. D: $\boxed{\text{All are true}}$. The first two are well-known. To show the third, note that $\lim_{n\to\infty}\int_a^b f(x)g(nx)dx=\lim_{n\to\infty}\int_a^b f(x)\left(g(nx)-\frac{1}{p}\int_0^p g(t)dt+\frac{1}{p}\int_0^p g(t)dt\right)dx=\lim_{n\to\infty}\left(\int_a^b f(x)dx\right)\left(\frac{1}{p}\int_0^p g(t)dt\right)+\int_a^b f(x)\left(g(nx)-\frac{1}{p}\int_0^p g(t)dt\right)dx$. Let $h(nx)=g(nx)-\frac{1}{p}\int_0^p g(t)dt$, and note that $\int_0^{\frac{p}{n}} h(nx)dx=\int_0^{\frac{p}{n}} g(nx)-\frac{1}{p}\int_0^p g(t)dt\,dx=$

$\int_0^{\frac{p}{n}} g(nx)dx - \frac{1}{n}\int_0^p g(t)dt = \frac{1}{n}\int_0^p g(u)du - \frac{1}{n}\int_0^p g(t)dt = 0$. Therefore over any interval $[a,b]$, $h(nx)$ will be zero except perhaps for some leftover portion where the period $\frac{p}{n}$ does not divide into the interval. The maximum value of this leftover portion would be $\max\big(h(x)\big)\frac{p}{n} \to 0$ as $n \to \infty$. Therefore the integral of $h(nx)$ is zero over any interval as $n \to \infty$, and hence $\lim_{n\to\infty}\int_a^b f(x)h(nx)dx = 0$ because $f(x)$ is bounded. This what remains is $\lim_{n\to\infty}\left(\int_a^b f(x)dx\right)\left(\frac{1}{p}\int_0^p g(t)dt\right) = \left(\int_a^b f(x)dx\right)\left(\frac{1}{p}\int_0^p g(t)dt\right)$.

26. C: From the previous question III with $f(x) = x^2$, $\lim_{n\to\infty}\int_0^1 \frac{x^2}{(1+|\sin(nx)|)^2}dx =$ $\left(\int_0^1 x^2 dx\right)\left(\frac{1}{\pi}\int_0^\pi \frac{dt}{(1+\sin(t))^2}\right) = \frac{1}{3\pi}\int_0^\pi \frac{dt}{(1+\sin(t))^2} = \frac{1}{3\pi}\int_0^\pi \frac{(1-\sin(t))^2}{(1-\sin^2(t))^2}dt = \frac{1}{3\pi}\int_0^\pi \frac{1-2\sin(t)+\sin^2(t)}{\cos^4(t)}dt =$ $\frac{1}{3\pi}\int_0^\pi \frac{1-2\sin(t)+\sin^2(t)}{\cos^4(t)}dt = \frac{1}{3\pi}\frac{4}{3} = \boxed{\frac{4}{9\pi}}$ (This can also be done with tangent half angle substitution).

27. B: $f(x) = \sum_{k=0}^{2025}\left(\sin|x-k| + \cos\left|x-k+\frac{1}{2}\right|\right) \to$ $f'(x) = \sum_{k=0}^{2025}\left(\cos|x-k|\cdot\frac{|x-k|}{x-k} - \sin\left|x-k+\frac{1}{2}\right|\cdot\frac{\left|x-k+\frac{1}{2}\right|}{x-k+\frac{1}{2}}\right)$. Obviously, potential discontinuities in $f'(x)$ will occur at the various absolute values. However, only integers actually cause a problem. To see this, note that if we consider the limit of the portion $\lim_{x\to\frac{2k-1}{2}}\left(\sin\left|x-\frac{2k-1}{2}\right|\cdot\frac{\left|x-\frac{2k-1}{2}\right|}{x-\frac{2k-1}{2}}\right) = \lim_{x\to\frac{2k-1}{2}}\left(\frac{\sin\left|x-\frac{2k-1}{2}\right|}{\left|x-\frac{2k-1}{2}\right|}\cdot\left(x-\frac{2k-1}{2}\right)\right) = \lim_{x\to\frac{2k-1}{2}}\left(1\cdot\left(x-\frac{2k-1}{2}\right)\right) = 0$. On the other hand, $\lim_{x\to k}\left(\cos|x-k|\cdot\frac{|x-k|}{x-k}\right) = \lim_{x\to k}\left(1\cdot\frac{|x-k|}{x-k}\right)$ does not exist. So the answer is the number of integers in the sum, $\boxed{2026}$.

28. A:

$\int_0^\infty e^{-\left(\frac{x^2}{4}+\frac{2025}{x^2}\right)}dx = e^{-45}\int_0^\infty e^{-\left(\frac{x}{2}-\frac{45}{x}\right)^2}dx$. Let $u = \frac{90}{x} \to x = \frac{90}{u} \to dx = -\frac{90}{u^2}$. So $e^{-45}\int_0^\infty e^{-\left(\frac{x}{2}-\frac{45}{x}\right)^2}dx =$ $e^{-45}\int_0^\infty e^{-\left(\frac{45}{u}-\frac{u}{2}\right)^2}\left(\frac{90}{u^2}\right)du$. Therefore $2e^{-45}\int_0^\infty e^{-\left(\frac{x}{2}-\frac{45}{x}\right)^2}dx = e^{-45}\int_0^\infty e^{-\left(\frac{x}{2}-\frac{45}{x}\right)^2}dx +$ $e^{-45}\int_0^\infty e^{-\left(\frac{45}{x}-\frac{x}{2}\right)^2}\left(\frac{90}{x^2}\right)dx = e^{-45}\int_0^\infty e^{-\left(\frac{x}{2}-\frac{45}{x}\right)^2}\left(1+\frac{90}{x^2}\right)dx = 2e^{-45}\int_0^\infty e^{-\left(\frac{x}{2}-\frac{45}{x}\right)^2}\left(\frac{1}{2}+\frac{45}{x^2}\right)dx =$ $2e^{-45}\int_0^\infty e^{-(u)^2}du = 2e^{-45}\sqrt{\pi}$. So $\int_0^\infty e^{-\left(\frac{x}{2}-\frac{45}{x}\right)^2}dx = \boxed{e^{-45}\sqrt{\pi}}$.

29. A: Note that $3+\cos^2(x)+\sin^4(x) - (3+\sin^2(x)+\cos^4(x)) = cos^2(x) - \cos^4(x) - \sin^2(x) + \sin^4(x) = \cos^2(x)\,(1-\cos^2(x)) - \sin^2(x)\,(1-\sin^2(x)) = \cos^2(x)\sin^2(x) - \cos^2(x)\sin^2(x) = 0$. This means these are the same value and $\cos(3+\sin^2(x)+\cos^4(x)) = \pm\sqrt{1-\sin^2(3+\cos^2(x)+\sin^4(x))} =$ $\pm\sqrt{1-\left(\frac{24}{25}\right)^2} = \pm\sqrt{\frac{25^2-24^2}{25^2}} = \pm\frac{7}{25}$.

The key part is determining the sign. Since $3+\sin^2(x)+\cos^4(x) = 4-\cos^2(x)+\cos^4(x)$. Let $f(x) = 4 - x + x^2$ for $x \in [0,1]$. Then the range of this function on this interval corresponds to the possible values of the argument of the answer. Clearly $f'(x) = -1+2x \to x = \frac{1}{2} \to f\left(\frac{1}{2}\right) = 3.75$. Also, $f(0) = f(1) = 4$. Since $\pi \le 3.75, 4 < \frac{3\pi}{2}$, this value is in Quadrant III with negative cosine so the answer is $\boxed{-\frac{7}{25}}$

30. B: This is the area of a semicircle with radius 45. $\boxed{\frac{2025\pi}{2}}$.