ANSWER KEY:

1. **B**
2. **C**
3. **D**
4. **A**
5. **C**
6. **D**
7. **D**
8. **D**
9. **A**
10. **C**
11. **B**
12. **B**
13. **D**
14. **A**
15. **E**
16. **D**
17. **A**
18. **C**
19. **D**
20. **B**
21. **D**
22. **E**
23. **A**
24. **B**
25. **D**
26. **B**
27. **C**
28. **E**
29. **D**
30. **B**

SOLUTIONS:

1. $\boxed{B}$: odd function $\to \boxed{\mathbf{0}}$

2. $\boxed{C}$: $\displaystyle\int_0^2 (x^3+3x^2+5x+2)\,dx = \left(\tfrac{16}{4}+8+\tfrac{5\cdot 4}{2}+4\right)-0 = 4+8+10+4 = \boxed{\mathbf{26}}$

3. $\boxed{D}$: $\displaystyle\int_{-1}^{1}(20252025\,x^{2025}+1)\sqrt{1-x^2}\,dx = 20252025\underbrace{\int_{-1}^{1} x^{2025}\sqrt{1-x^2}\,dx}_{\text{odd}\Rightarrow 0} + \underbrace{\int_{-1}^{1}\sqrt{1-x^2}\,dx}_{\text{area of unit semicircle}} = 0+\frac{\pi}{2} = \boxed{\frac{\mathbf{\Pi}}{\mathbf{2}}}.$

4. $\boxed{A}$: $\displaystyle I = \int_0^{\pi/2}\ln(\tan x)\,dx \xrightarrow{\frac{\pi}{2}-x=x} = \int_0^{\pi/2}\ln(\cot x)\,dx = \int_0^{\pi/2}\ln\left(\tfrac{1}{\tan x}\right)dx = -I \to I=-I \to \boxed{\mathbf{0}}$

5. $\boxed{C}$: $\displaystyle\int_0^1 e^{x+e^x}\,dx \xrightarrow{u=e^x} \int_1^e e^u\,du = \left[e^u\right]_1^e = \boxed{\mathbf{e^e-e}}$

6. $\boxed{D}$: $\displaystyle\int_0^{\sqrt{2}}\frac{x}{x^4+4}\,dx \xrightarrow{t=x^2} \frac{1}{2}\int_0^2\frac{dt}{t^2+4} = \frac{1}{4}\left[\arctan\left(\frac{t}{2}\right)\right]_0^2 = \boxed{\frac{\mathbf{\Pi}}{\mathbf{16}}}$

7. $\boxed{D}$: $\displaystyle\int_0^{\pi}\sin x\,\sin 2x\,dx = \frac{1}{2}\int_0^{\pi}(\cos x-\cos 3x)\,dx = \frac{1}{2}\left[\sin x-\tfrac{\sin 3x}{3}\right]_0^{\pi} = \boxed{\mathbf{0}}$

8. $\boxed{D}$: $\displaystyle\int_0^{\pi/2}\frac{\sin x}{\cos x+1}\,dx \xrightarrow{u=\cos(x)} \int_1^0\frac{-\,du}{u+1} = \int_0^1\frac{du}{u+1} = \left[\ln(u+1)\right]_0^1 = \boxed{\ln\mathbf{2}}$

9. $\boxed{A}$: $\displaystyle\int_0^{\infty}\frac{\arctan x}{x^2+1}\,dx \xrightarrow{x=\tan t} \int_0^{\pi/2} t\,dt = \left[\tfrac{t^2}{2}\right]_0^{\pi/2} = \boxed{\frac{\mathbf{\Pi^2}}{\mathbf{8}}}$

10. $\boxed{C}$: The value inside the root is 3 $\to \displaystyle\int_0^{\frac{\pi}{3}}\sqrt{3}\,dx = \sqrt{3}\left[x\right]_0^{\pi/3} = \boxed{\frac{\mathbf{\Pi\sqrt{3}}}{\mathbf{3}}}$

11. $\boxed{B}$: $\displaystyle\int_0^{\infty}\frac{dx}{e^x+1} \xrightarrow{u=e^{-x}} \int_0^1\frac{du}{1+u} = \left[\ln|1+u|\right]_0^1 = \boxed{\ln\mathbf{2}}$

12. $\boxed{B}$: $\displaystyle\int_0^{\infty}\frac{x^2}{(x^2+1)^2}\,dx = \int_0^{\infty}\left(\frac{1}{x^2+1}-\frac{1}{(x^2+1)^2}\right)dx \xrightarrow{x=\tan t} \int_0^{\frac{\pi}{2}} 1-\cos^2 t\,dt = \boxed{\frac{\mathbf{\Pi}}{\mathbf{4}}}$

13. $\boxed{D}$: $\displaystyle\int_0^{\pi}\frac{(x+1)\sin x}{1+\sin x}\,dx \xrightarrow{x=\pi-x} \int_0^{\pi}\frac{(\pi-x+1)\sin x}{1+\sin x}\,dx = \frac{1}{2}\int_0^{\pi}\frac{\left[(x+1)+(\pi-x+1)\right]\sin x}{1+\sin x}\,dx =$

$\displaystyle\frac{\pi+2}{2}\int_0^{\pi}\frac{\sin x}{1+\sin x}\,dx = \frac{\pi+2}{2}\int_0^{\pi} 1-\left(\sec^2 x-\frac{\sin x}{\cos^2 x}\right)dx = \frac{\pi+2}{2}\left[x-\tan x+\sec x\right]_0^{\pi} = \frac{\pi+2}{2}(\pi-2) \to \boxed{\frac{\mathbf{\Pi^2-4}}{\mathbf{2}}}.$

14. $\boxed{\text{A}}$: $\displaystyle\int_1^2 \frac{2x^9+x^6+x^4}{(x^8+x^5+3x^3)^2}\,dx = \int_1^2 \frac{2x^3+1+x^{-2}}{(x^5+x^2+3)^2}\,dx = \int_1^2 \frac{2x^5+x^2+1}{(x^6+x^3+3x)^2}\,dx \xrightarrow{u=x^6+x^3+3x} \int_5^{78} \frac{1}{3u^2}\,du = \frac{73}{1170} \to$
$1170 \mod 73 = 16\cdot 73+2 \mod 73 \equiv \boxed{2}$

15. $\boxed{\text{E}}$: $\displaystyle\int_0^3 \frac{x^{2025}}{-18-3x+13x^2+7x^3+x^4}\,dx = \int_0^3 \frac{x^{2025}}{(x-1)(x+2)(x+3)^2} \to \boxed{\textbf{Divergent}}$

16. $\boxed{\text{D}}$: $\displaystyle\int_0^1 \ln\Gamma(x)\,dx = \int_0^1 \ln\left(\frac{\pi\csc(\pi x)}{\Gamma(1-x)}\right)dx = \tfrac{1}{2}\int_0^1 [\ln\pi - \ln\sin(\pi x)]\,dx = \tfrac{1}{2}(\ln\pi+\ln 2) = \boxed{\frac{\mathbf{1}}{\mathbf{2}}\ln(\mathbf{2\Pi})}$

17. $\boxed{\text{A}}$: If $\displaystyle f(x) = \frac{1}{\Gamma(x)\left[\Gamma(x)+\Gamma(1-x)\right]}$, then $\displaystyle f(x)+f(1-x) = \frac{\Gamma(x)+\Gamma(1-x)}{\Gamma(x)\Gamma(1-x)\left[\Gamma(x)+\Gamma(1-x)\right]} = \frac{1}{\Gamma(x)\Gamma(1-x)} =$
$\displaystyle\frac{\sin(\pi x)}{\pi}$, therefore k$\displaystyle I = \int_0^1 \frac{\sin(\pi x)}{2\pi}\,dx = \frac{1}{2\pi}\left[-\frac{\cos(\pi x)}{\pi}\right]_0^1 = \boxed{\frac{\mathbf{1}}{\mathbf{\Pi^2}}}$

18. $\boxed{\text{C}}$: $\displaystyle\int_1^9 \left(\sqrt[3]{x-\sqrt{x^2-1}} + \sqrt[3]{x+\sqrt{x^2-1}}\right)dx \xrightarrow{t=x-\sqrt{x^2-1}} \frac{1}{2}\int_1^{9-4\sqrt{5}} \left(t^{\frac{1}{3}}+t^{-\frac{1}{3}}\right)\left(1-\frac{1}{t^2}\right)dt =$
$\displaystyle\frac{1}{2}\int_1^{9-4\sqrt{5}} \left(t^{\frac{1}{3}}+t^{-\frac{1}{3}}-t^{-\frac{5}{3}}-t^{-\frac{7}{3}}\right)dt. = \left[\tfrac{3}{4}t^{\frac{4}{3}}+\tfrac{3}{2}t^{\frac{2}{3}}-\tfrac{3}{4}t^{-\frac{2}{3}}-\tfrac{3}{8}t^{-\frac{4}{3}}\right]_1^{9-4\sqrt{5}} = \frac{165}{8} \to 165+8 = \boxed{173}$
Alternatively, consider $m+n = y, m^3+n^3 = 2x, mn = 1, (m+n)(m^2-mn+n^2) = (m+n)((m+n)^2-3mn) = y(y^2-3) = 2x$, solve the integral from here by using the inverse.

19. $\boxed{\text{D}}$: $\displaystyle\int_3^7 \frac{\ln(x+2)}{\ln[(12-x)(x+2)]}\,dx \xrightarrow{x=10-x} \frac{1}{2}\int_3^7 \frac{\ln(x+2)+\ln(12-x)}{\ln[(12-x)(x+2)]}\,dx = \frac{1}{2}\int_3^7 1\,dx = \boxed{2}$

20. $\boxed{\text{B}}$: $\displaystyle\int_0^{\ln 2} \arctan(1+e^x)\,dx \xrightarrow{t=e^x} I(a) = \int_1^2 \frac{\arctan(1+at)}{t}\,dt \to I'(a) = \int_1^2 \frac{dt}{1+(1+at)^2} = \frac{\arctan(1+2a)-\arctan(1+a)}{a}$
and $\displaystyle I(0) = \frac{\pi\ln 2}{4} \Rightarrow I(1) = I(0)+\int_0^1 I'(a)\,da = \boxed{\frac{\mathbf{3\Pi\ln 2}}{\mathbf{8}}}$
Alternatively, substitute $u = \ln(2)-x$ and add. Recall the identity $\arctan(a)+\arctan(b) = \arctan(\frac{a+b}{1-ab})$. If we do $\arctan(1+e^x)+\arctan(1+e^{\ln 2-x})$, $\frac{a+b}{1-ab} = -1$. Since $1+e^x > 0, 1+e^{\ln(2)-x} > 0$, the sum is $-\frac{\pi}{4}+\pi = \frac{3\pi}{4}$. The desired value is $\frac{3\pi}{4}\cdot\ln(2)\cdot\frac{1}{2} = \boxed{\frac{\mathbf{3\Pi\ln 2}}{\mathbf{8}}}$

21. $\boxed{\text{D}}$: $\displaystyle\int_0^\infty \frac{1}{2^{2^{\lfloor x\rfloor}}+\{x\}}dx = \sum_{n=0}^\infty \int_n^{n+1} \frac{dx}{2^{2^n}+x-n} = \sum_{n=0}^\infty \int_0^1 \frac{du}{2^{2^n}+u} = \sum_{n=0}^\infty \ln\left(\frac{1+2^{2^n}}{2^{2^n}}\right) = \ln\left(\prod_{n=0}^\infty \left(1+2^{-2^n}\right)\right)$
$\displaystyle\prod_{n=0}^\infty \left(1+x^{2^n}\right) = (1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^k}) = \frac{1-x^{2^{k+1}}}{1-x} \to \lim_{k\to\infty} \frac{1-\left(\frac{1}{2}\right)^{2^{k+1}}}{1-\frac{1}{2}} = 2 \Rightarrow \boxed{\ln\mathbf{2}}$

22. $\boxed{\text{E}}$: $\displaystyle\int_0^1 \frac{\arctan x}{x\sqrt{1-x^2}}\,dx \xrightarrow{x=\sin\theta} F(a) = \int_0^{\pi/2} \frac{\arctan(a\sin\theta)}{\sin\theta}\,d\theta \to F'(a) = \int_0^{\pi/2} \frac{d\theta}{1+a^2\sin^2\theta} = \frac{\pi}{2\sqrt{1+a^2}}$ and $F(0) =$
$\displaystyle 0 \to F(a) = \frac{\pi}{2}\sinh^{-1}a \Rightarrow F(1) = \boxed{\frac{\mathbf{\Pi}}{\mathbf{2}}\ln\left(\mathbf{1}+\sqrt{\mathbf{2}}\right)}$.

Alternatively, $I(\alpha)=\int_0^1 \frac{\arctan \alpha x}{x\sqrt{1-x^2}}\,dx$, $I'(\alpha)=\int_0^1 \frac{1}{(1+(\alpha x)^2)\sqrt{1-x^2}}dx; x=\sin(\theta) \implies \int_0^{\frac{\pi}{2}} \frac{1}{1+(\alpha\sin\theta)^2}d\theta.$
Factor $\sin^2(\theta)$ from both the numerator and the denominator and apply the Pythagorean identity.

23. $\boxed{\text{A}}$: $\int_0^\infty \frac{x^4}{(x^4+x^2+1)^3}\,dx=\int_0^\infty \frac{dx}{x^2\left[\left(x+\frac{1}{x}\right)^2-1\right]^3} \xrightarrow{u=x+\frac{1}{x}} \frac{1}{2}\int_2^\infty \frac{u\,du}{(u^2-4)^4}=\boxed{\frac{\Pi}{48\sqrt{3}}}$

24. $\boxed{\text{B}}$: $\int_0^{\pi/2} \ln^2(\tan x)\,dx=\int_0^{\frac{\pi}{2}}\left[-2\sum_{n=0}^\infty \frac{\cos(2(2n+1)x)}{2n+1}\right]^2 dx=4\sum_{n=0}^\infty \frac{1}{(2n+1)^2}\int_0^{\pi/2}\cos^2(2\,(2n+1)\,x)\,dx=\pi\sum_{n=0}^\infty \frac{1}{(2n+1)^2}=$
$\pi\cdot\frac{\pi^2}{8}=\boxed{\frac{\Pi^3}{8}}$
Or $\tan(x)=u \to \int_0^\infty \frac{\ln(x)^2}{1+x^2}dx \xrightarrow{x=e^u} \int_{-\infty}^\infty \frac{u^2e^u}{1+e^{2u}}du=\int_{-\infty}^\infty \frac{u^2e^{-u}}{1+e^{-2u}}du=2\int_0^\infty \frac{u^2e^{-u}}{1+e^{-2u}}du=2\int_0^\infty u^2\sum_{n=0}^\infty e^{-(2n+1)u}du=$
$4\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^3}=\boxed{\frac{\Pi^3}{8}}$

25. $\boxed{\text{D}}$ $I=\int_0^{\pi/2}\frac{\sin x+2\cos x}{1+\sqrt{\sin 2x}}\,dx \xrightarrow{x\mapsto\frac{\pi}{2}-x} 2I=\int_0^{\pi/2}\frac{3(\sin x+\cos x)}{1+\sqrt{\sin 2x}}\,dx=\int_0^{\pi/2}\frac{3\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}{1+\sqrt{\sin 2x}}\,dx=3J$. Let $x=\frac{\pi}{4}+\theta$,
then $J=\int_0^{\pi/2}\frac{\sin\left(x+\frac{\pi}{4}\right)}{1+\sqrt{\sin 2x}}\,dx=2\sqrt{2}\int_0^{\pi/4}\frac{\cos\theta}{1+\cos\theta}\,d\theta=2\int_0^{\pi/2}\frac{\cos\theta}{1+\cos\theta}\,d\theta=\pi-2\to I=\boxed{\frac{3(\Pi-2)}{2}}$

26. $\boxed{\text{B}}$ $I(\alpha)=\int_0^1\frac{x\ln(\alpha+x)}{1+x^2}\,dx.$ $I'(\alpha)=\int_0^1\frac{x}{(1+x^2)(\alpha+x)}\,dx=\frac{1}{1+\alpha^2}\int_0^1\left[\frac{\alpha x+1}{1+x^2}-\frac{\alpha}{\alpha+x}\right]dx=\frac{\alpha}{2(1+\alpha^2)}\ln 2+$
$\frac{\pi}{4(1+\alpha^2)}-\frac{\alpha}{1+\alpha^2}\ln(1+\alpha)\to I(\alpha)=\frac{1}{8}\left(\ln\!\left(1+\alpha^2\right)\ln 2+\pi\arctan\alpha\right)+C$
$I(0)=\int_0^1\frac{x\ln x}{1+x^2}dx=\int_0^1 x\ln x\sum_{n=0}^\infty(-1)^n x^{2n}dx=\sum_{n=0}^\infty(-1)^n\int_0^1 x^{2n+1}\ln x\,dx=\sum_{n=0}^\infty(-1)^n\left[-\frac{1}{(2n+2)^2}\right]=-\sum_{n=0}^\infty\frac{(-1)^n}{(n+1)^2}=$
$-\sum_{m=1}^\infty\frac{(-1)^{m-1}}{m^2}=-\frac{\pi^2}{12}\cdot\frac{1}{4}=-\frac{\pi^2}{48}\to(I(1)=\frac{1}{8}\left(\ln^2 2+\frac{\pi^2}{4}\right)-\frac{\pi^2}{48}=\frac{\ln^2 2}{8}+\frac{\pi^2}{96}\to 12+2+1+96=\boxed{111}$

27. $\boxed{\text{A}}$: $\lim_{n\to\infty}\left\{n\int_0^{\pi/4}e^x\tan^n x\,dx\right\}\xrightarrow{x=\frac{\pi}{4}-\frac{t}{n}}\lim_{n\to\infty}\int_0^{n\pi/4}e^{\pi/4-t/n}\left(\frac{1-\frac{t}{n}}{1+\frac{t}{n}}\right)^n dt\to e^{\pi/4}\int_0^\infty e^{-2t}\,dt=\frac{e^{\pi/4}}{2}=\boxed{\frac{\mathbf{e}^{\Pi/4}}{2}}$

28. $\boxed{\text{E}}$: $\lim_{n\to\infty}\left\{n\int_0^1\frac{x^n}{x^3+1}\,dx\right\}\xrightarrow{u=x^n}\int_0^1\frac{u^{\frac{1}{n}}}{1+u^{\frac{3}{n}}}\,du\xrightarrow{n\to\infty}\int_0^1\frac{1}{2}\,du=\boxed{\frac{1}{2}}$

29. $\boxed{\text{D}}$: $\int_0^1\frac{x\sqrt{x}\ln(x)}{x^2-x+1}\,dx\xrightarrow{x=u^2}\int_0^1\frac{4u^4\ln(u)}{u^4-u^2+1}\,du=4\int_0^1\frac{u^4(u^2+1)\ln(u)}{u^6+1}\,du=4\int_0^1\left(\ln(u)-\frac{\ln(u)}{u^6+1}+\frac{u^4\ln(u)}{u^6+1}\right)du$
Note $\int_0^1\frac{u^4\ln(u)}{u^6+1}du\xrightarrow{v=\frac{1}{u}}-\int_1^\infty\frac{\ln(v)}{v^6+1}\,dv\xrightarrow{v^6=t}-\frac{1}{36}\int_0^\infty\frac{t^{-\frac{5}{6}}\ln(t)}{t+1}dt\to I(\alpha)=\int_0^\infty\frac{t^{-\alpha}}{t+1}dt\to I'(\alpha)=\int_0^\infty-\frac{\ln(t)t^{-\alpha}}{t+1}dt\to$

$$I'\left(\frac{5}{6}\right) \xrightarrow{\beta} I(\alpha) = \Gamma(t)\Gamma(1-t) = \pi\csc\pi t \to I'(\alpha) = \frac{-\pi^2\cos(\pi t)}{(\sin(\pi t)^2} \xrightarrow{t=\frac{5}{6}} -2\sqrt{3}\pi^2 \to 4(-1+\frac{2\sqrt{3}\pi^2}{36}) = \boxed{\mathbf{\frac{2\sqrt{3}\Pi^2}{9} - 4}}$$

30. $\boxed{\textbf{B}}$: Recall the identity of $\phi - 1 = \frac{1}{\phi}$. Write the integral to

$$\int_0^\infty \frac{x^\phi \arctan(x)}{x(x^\phi+1)^2}dx = \left(\int_0^1 + \int_1^\infty\right)\frac{x^\phi \arctan(x)}{x(x^\phi+1)^2}dx$$

.
Note

$$\int_1^\infty \frac{x^\phi \arctan(x)}{x(x^\phi+1)^2}dx = \int_1^0 \frac{\frac{1}{x^\phi}(\frac{\pi}{2}-\arctan(x)}{(x^{-\phi}+1)^2}\cdot x\cdot\frac{-1}{x^2}dx = \int_0^1 \frac{x^\phi(\frac{\pi}{2}-\arctan(x)}{(x^\phi+1)^2\cdot x}dx$$

So the desired value is

$$\frac{\pi}{2}\int_0^1 \frac{x^\phi}{x(1+x^\phi)^2}dx \xrightarrow{x^\phi=u} \frac{\pi}{2\phi}\int_0^1\frac{1}{(1+x)^2}dx = \boxed{\mathbf{\frac{\Pi}{4\Phi}}}$$