Answer Key:

1. **C**
2. **C**
3. **A**
4. **D**
5. **D**
6. **D**
7. **A**
8. **C**
9. **E** – 1
10. **C**
11. **A**
12. **C**
13. **C**
14. **D**
15. **D**
16. **B**
17. **D**
18. **B**
19. **A**
20. **D**
21. **B**
22. **C**
23. **B**
24. **B**
25. **A**
26. **D**
27. **E** – 5
28. **E** – 5416
29. **D**
30. **B**

Solutions:

1. $\boxed{\text{C}}$: The negation of a conditional statement of the form "If P then Q" is "P and not Q." So the negation is "Nylah is a black belt and Daniela is not a brown belt."

2. $\boxed{\text{C}}$: The negation of an existential statement concerning the property P is a universal statement about the property $\neg P$. Thus, the negation is "For all real numbers x, x is rational."

3. $\boxed{\text{A}}$: This statement implies that for any person x we may possibly choose, there is another person y which makes x older than y. So every person is older than some other person; everybody is older than somebody.

4. $\boxed{\text{D}}$: The only row which gives a true result is when P is false and Q is true. If P is false, then $\neg P$ is true. Hence, we want $\neg P \wedge Q$ to be true.

5. $\boxed{\text{D}}$: The binary representation of 2025 is 11111101001, which has 8 ones.

6. $\boxed{\text{D}}$: Since $k-1 < k - \frac{1}{2} < k$, $\lceil k - \frac{1}{2} \rceil = k$ and $\lfloor k - \frac{1}{2} \rfloor = k-1$. Then the sum is $2k-1$.

7. $\boxed{\text{A}}$: By the Euclidean algorithm, in each step after the first, we take the previous divisor and the previous remainder and apply the quotient-remainder theorem. Thus, with a divisor of 611 and a remainder of 286, we get $611 = 286 \cdot 2 + 39$.

8. $\boxed{\text{C}}$: In the first pass through the loop, $a = 2$ and $i = 1$, so that $\frac{a}{2} + \frac{1}{a} = 1 + \frac{1}{2} = \frac{3}{2}$ becomes the new value of a. In the second pass, $a = \frac{3}{2}$ and $i = 2$, so that

$$\frac{3/2}{2} + \frac{2}{3/2} = \frac{3}{4} + \frac{4}{3} = \frac{25}{12}$$

becomes the new value of a. In the third and final pass, $a = \frac{25}{12}$ and $i = 3$, so we get

$$\frac{25/12}{2} + \frac{3}{25/12} = \frac{25}{24} + \frac{36}{25} = \frac{1489}{600}.$$

Therefore $m = 1489$ and $n = 600$ and thus $1489 \bmod 600 \equiv 289$.

9. $\boxed{\text{E}}$: There are 26 choices for each position in the six-letter string. This is 26^6 possible strings. By Fermat's Little Theorem, $26^6 \bmod 7 \equiv 1$.

10. $\boxed{\text{C}}$: Partition the triangle into smaller equilateral triangles of side length $\frac{1}{12}$. There are $12 + 2(11 + 10 + \cdots + 1) = 2 \cdot 12 \cdot 13/2 - 12 = 144$ such triangles. Place a point within each triangle. Then by the pigeonhole principle, the next point placed will necessarily be within $\frac{1}{12}$ of an inch from another point. We therefore need 145 points.

11. $\boxed{\textbf{A}}$: We can choose the 8 students who receive the correct trophies in $\binom{10}{8}$ ways. The remaining two students must have their trophies swapped in order to receive the incorrect trophies, which happens in only one way. Hence, $N = \binom{10}{8} = 45$.

12. $\boxed{\textbf{C}}$: By Fermat's Little Theorem, $2^{42} \equiv 1 \bmod 43$. Since $2^{2025} = (2^{42})^{48} \cdot 2^9$, we have that $2^{2025} \equiv 2^9 \bmod 43$. Noting that $2^9 = 512 = 43 \cdot 11 + 39$, we arrive at $2^{2025} \equiv 2^9 \equiv 39 \bmod 43$.

13. $\boxed{\textbf{C}}$: From $5x + 3 \equiv 2 \bmod 11$ we have $5x \equiv -1 \bmod 11 \equiv 10 \bmod 11$. Hence, $x \equiv 2 \bmod 11$. This implies that 11 divides $x - 2$, so there is an integer p such that $11p = x - 2$. From $6x + 1 \equiv 3 \bmod 7$ we have $6x \equiv 2 \bmod 7$, which gives $3x \equiv 1 \bmod 7$. Similarly, there is an integer q such that $7q = 3x - 1$. Substituting gives $7q = 3(11p + 2) - 1 = 33p + 5$. Take the equation $7q - 33p = 5$ modulo 7 to get $2p \equiv 5 \bmod 7$. This has solution $p = 6$. It follows that $x = 11p + 2 = 66 + 2 = 68$. Since all solutions are congruent to 68 modulo the least common multiple of 11 and 7—which is 77—68 is the smallest positive solution. Thus, $n = 68 \bmod 9 \equiv 5$.

14. $\boxed{\textbf{D}}$: Noting the proximity of this number to $20^4 = 160000$, the number is $160000 - 2799 = 20^4 - 7 \cdot 20^2 + 1$. $x^4 - 7x^2 + 1 = (x^4 + 2x^2 + 1) - 9x^2 = (x^2 + 1)^2 - (3x)^2 = (x^2 + 3x + 1)(x^2 - 3x + 1)$. When $x = 20$, this is $461 \cdot 341 = 461 \cdot 31 \cdot 11$, so the number has $(1 + 1)(1 + 1)(1 + 1) = 8$ factors.

15. $\boxed{\textbf{D}}$: Both k and m must be integers greater than 5. Expanding each in terms of the bases, we obtain $3k^2+2k+5 = 3m+5$. This reduces to $3k^2+2k = k(3k+2) = 3m$. From $k \geq 6$ it follows that $3k+2 \geq 20$. The smallest value of $k(3k+2)$ is therefore $6 \cdot 20 = 120$, which implies the smallest value of m is $m = 120/3 = 40$.

16. $\boxed{\textbf{B}}$: Let A be the event that all the flips are tails and B the condition that one is a tail. Then $P(B) = 1 - \frac{1}{1024}$ since the one flip being a tail is the complement of all flips being heads. Also, $P(A \cap B)$ is the probability that all the flips are tails, which is $\frac{1}{1024}$. Hence,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{1024}}{1 - \frac{1}{1024}} = \frac{1}{1024 - 1} = \frac{1}{1023}.$$

17. $\boxed{\textbf{D}}$: We use the principle of inclusion-exclusion to subtract the number of integers divisible by $\gcd(12, 18) = 36$ from the sum of the number divisible by 12 and the number divisible by 18. We get

$$\left\lfloor \frac{1000}{12} \right\rfloor + \left\lfloor \frac{1000}{18} \right\rfloor - \left\lfloor \frac{1000}{36} \right\rfloor = \left\lfloor \frac{250}{3} \right\rfloor + \left\lfloor \frac{500}{9} \right\rfloor - \left\lfloor \frac{250}{9} \right\rfloor = 83 + 55 - 27 = 111.$$

18. $\boxed{\textbf{B}}$: To be divisible by 33, $28577d245$ must be divisible by both 3 and 11. The sum of the digits is $d + 40$ and the alternating sum of the digits is $2 - d$. Since we need d to be a digit and 3 to divide $d + 40$ and 11 to divide $2 - d$, d must be 2.

19. $\boxed{\textbf{A}}$: Note that both 2464 and 2947 are divisible by 7, so the equation becomes $352a + 421b = 1$. Now we use the Euclean algorithm to determine a and b. We have

$$\begin{aligned} 421 &= 352 \cdot 1 + 69 \\ 352 &= 69 \cdot 5 + 7 \\ 69 &= 7 \cdot 9 + 6 \\ 7 &= 6 \cdot 1 + 1. \end{aligned}$$

Now we back substitute:

$$\begin{aligned} 1 &= 7 - 6 \\ &= 7 - (69 - 7 \cdot 9) = 7 \cdot 10 - 69 \\ &= (352 - 69 \cdot 5) \cdot 10 - 69 = 352 \cdot 10 - 69 \cdot 51 \\ &= 352 \cdot 10 - (421 - 352) \cdot 51 = 352 \cdot 61 - 421 \cdot 51 \\ &= 352 \cdot 61 + 421 \cdot (-51). \end{aligned}$$

Hence, $a + b = 61 - 51 = 10$.

20. $\boxed{\textbf{D}}$: The coefficient is $\dfrac{7!}{3!3!} = 7 \cdot 5 \cdot 4 = 140$.

21. $\boxed{\textbf{B}}$: $\dfrac{10!}{3!4!} = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 5 = 5040 \cdot 5 = 25200$.

22. $\boxed{\textbf{C}}$: We use the method of sticks and stones, where the stones represent the 10 two-liters and we have 5 sticks that separate the stones into six piles. However, since there must be at least one Diet Fizz, at least two must be Dr. Fizz, and at least three must be Fizz Cola, we drop 1, 2, and 3 stones into these piles. This leaves us 4 stones to distribute with 5 sticks. To take care of the fact that we can have at most one can be Lemon Fizz, we find the number of ways to select no Lemon Fizzes and exactly 1 Lemon Fizz. For the number of ways to get no Lemon Fizzes, we remove a stick; this is $\binom{4+4}{4} = \binom{8}{4} = 70$. For the number of ways to get 1 Lemon Fizz, we drop 1 stone in that pile and remove a stick, leaving 3 stones and 4 sticks: $\binom{3+4}{4} = \binom{7}{4} = 35$. The sum is 105.

23. $\boxed{\textbf{B}}$: There is only 1 false statement, Statement II. (A counterexample to Statement II is to let the irrational number be $\sqrt{2}$ and the rational number be 0. Then $0 \cdot \sqrt{2} = 0$ which is rational.)

24. $\boxed{\textbf{B}}$: A common theorem in graph theory states that a connected tree on n vertices has $n - 1$ edges. Since T has 20 vertices, T has 19 edges.

25. $\boxed{\textbf{A}}$: The graph has many Hamiltonian paths. The graph does not have an Euler circuit because it has at least one vertex of odd degree. The graph does not have an Euler path because it has four vertices of odd degree (it must have only two vertices of odd degree and the rest even, or all even).

26. $\boxed{\textbf{D}}$: If a graph G has vertices of degrees 1, 2, 2, 3, and 3, then the sum of the degrees is 11, but the sum of the degrees must be even. So this graph does not exist.

27. $\boxed{\textbf{E}}$: Graph A is isomorphic to C; graph B is isomorphic to D; graph E is isomorphic to G; graph F is isomorphic to K; and graph I is isomorphic to J. There are 5 pairs.

28. $\boxed{\textbf{E}}$: This set is the complement of four-digit integers that contain no 4's or 7's. Because a four-digit integer cannot begin with a 0, the number of these is $7 \cdot 8 \cdot 8 \cdot 8 = 3584$. There are a total of 9000 four-digit integers, so the number of them with a 4 or 7 is $9000 - 3584 = 5416$.

29. $\boxed{\textbf{D}}$: To get the last two digits, we would normally take the product modulo 100, but these numbers are in base 6. So we take the product modulo 100_6. Thus,

$$\begin{aligned}(405 \cdot 11011)_6 \bmod 100_6 &\equiv (5 \cdot 11)_6 \bmod 100_6 \\ &\equiv 55_6 \bmod 100_6 \equiv (-1)_6 \bmod 100_6.\end{aligned}$$

It follows that, since 2025 is odd in any base, $((-1)_6)^{2025} \equiv (-1)_6 \bmod 100_6 \equiv 55_6 \bmod 100_6$. The last two digits are therefore 55.

30. $\boxed{\textbf{B}}$: Label the points from 1 to 14 and let a_n be the number of ways to join the points without intersecting segments, with $a_0 = 1$. Note that if we connect point 1 with point k, between them there must be an even number of point, so 1 must be connected with an even-numbered vertex. If 1 is connected with $2r$, then the $2r - 2$ points between 1 and $2r$ must be joined with points in that same section. The same holds for the other $14 - 2r$ points. Thus there are $a_{r-1}a_{7-r}$ ways to do this, for each $r = 1, 2, 3, 4, 5, 6, 7$. The total number of ways is the sum

$$\sum_{r=1}^{7} a_{r-1}a_{7-r} = \sum_{r=0}^{6} a_r a_{6-r}, \quad \text{with } a_0 = 1.$$

But this is the recurrence relation for the 7th Catalan number, C_7. Hence, the total number of ways to connect segments without intersections is $C_7 = \frac{1}{8}\binom{14}{7}$. Now, to connect the points arbitrarily, we can first choose 7 of them and then decide to which of the other 7 we will connect. This can be done in $\binom{14}{7}7!$ ways. However, we are counting each way to pair them 2^7 times (in the first 7 points we chose we needed one point of each pair). Thus there the total is $\binom{14}{7}7!/2^7$. Finally, the probability is

$$\frac{\frac{1}{8}\binom{14}{7}}{\binom{14}{7}\frac{7!}{2^7}} = \frac{2^7}{8 \cdot 7!} = \frac{2^4}{7!} = \frac{1}{7 \cdot 3 \cdot 5 \cdot 3} = \frac{1}{315}.$$

Hence, $m + n = 1 + 315 = 316$.