Answer Key:

1. C -

$$f(g(x)) = f(-x+10) = y(-x+10)+4 = -yx+10y+4$$

$$g(f(x)) = g(yx+4) = -(yx+4)+10 = -yx-4+10 = -yx+6$$
$$f(g(x)) = g(f(x)) \implies -yx+10y+4 = -yx+6$$
$$10y+4 = 6 \implies 10y = 2 \implies y = \frac{1}{5}$$
$$\boxed{\frac{1}{5}}$$

2. A -

$$f(x) = \frac{\ln(x-e)}{x^2-e^2}$$
$$\ln(x-e) \text{ is defined when } x > e$$
$$x^2-e^2 = (x-e)(x+e) \text{ is non-zero when } x \neq \pm e$$
$$x > e \quad \text{and} \quad x \neq e$$
$$\text{Domain: } (e, \infty)$$
$$\boxed{(e, \infty)}$$

3. D -

$$\log_2(x-1) + \log_4(x-4) = 2$$

$$x-1 > 0 \quad \text{and} \quad x-4 > 0 \implies x > 4$$

$$\log_4(x-4) = \frac{1}{2}\log_2(x-4)$$

$$\log_2(x-1) + \frac{1}{2}\log_2(x-4) = 2$$

$$2\log_2(x-1) + \log_2(x-4) = 4$$

$$\log_2\left((x-1)^2(x-4)\right) = 4$$

$$(x-1)^2(x-4) = 2^4 = 16$$

$$(x^2-2x+1)(x-4) = x^3-6x^2+9x-4 = 16$$

$$x^3-6x^2+9x-20 = 0$$

$$x = 5 \quad \text{is a real solution}$$

$$\log_2(5-1) + \log_4(5-4) = \log_2(4) + \log_4(1) = 2+0 = 2$$

$$\boxed{5}$$

4. B -

$$f(x) = \frac{x^3 - 8}{x^2 - 4}$$

Factor the numerator and denominator:

$$f(x) = \frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}$$

Cancel the common factor (x - 2):

$$f(x) = \frac{x^2+2x+4}{x+2}, \quad x \neq 2$$

Now, examine the asymptotes:

1. Vertical asymptote:

A vertical asymptote occurs where the denominator is zero and not canceled out.

$$x + 2 = 0 \implies x = -2$$

2. Oblique asymptote:

Since the degree of the numerator is higher than the degree of the denominator (degree 2 vs. degree 1), there is an oblique asy

$$x^2 + 2x + 4 \div (x+2)$$

The quotient is x with a remainder of 0, so the oblique asymptote is $y = x$.

Thus, there is one vertical asymptote at $x = -2$ and one oblique asymptote at $y = x$.

$\boxed{2}$

5. B - We are tasked with finding the coefficient of x^2 in the polynomial $Q_{15}(x)$, where the sequence of polynomials $Q_n(x)$ is defined recursively as $Q_n(x) = Q_{n-1}(x+n)$ for $n \geq 1$, and the initial polynomial is given by:

$$Q_0(x) = x^4 + 5x^3 - 12x^2 + 8x + 7.$$

The recursive formula $Q_n(x) = Q_{n-1}(x+n)$ shifts the argument of the polynomial $Q_{n-1}(x)$ by n. This means that every time we move from $Q_{n-1}(x)$ to $Q_n(x)$, we replace x with $x+n$ in the polynomial.

We are interested in tracking how the x^2 term changes under this recursion. In $Q_0(x)$, the x^2 term is $-12x^2$. Each time we apply the transformation $x \to x+n$, the binomial expansion will produce new terms involving powers of x, specifically from the expansion of $(x+n)^2$.

When we substitute $x+n$ for x, the x^2 term in $Q_{n-1}(x)$ changes as follows:

$$(x+n)^2 = x^2 + 2nx + n^2.$$

Thus, the term $-12x^2$ in $Q_0(x)$ becomes:

$$-12(x+n)^2 = -12(x^2 + 2nx + n^2) = -12x^2 - 24nx - 12n^2.$$

The new coefficient of x^2 remains -12, and the transformation does not affect the coefficient of x^2 as we iterate this process.

Since the transformation does not change the coefficient of x^2, the coefficient of x^2 in $Q_{15}(x)$ remains the same as in $Q_0(x)$, which is -12.

Thus, the coefficient of x^2 in $Q_{15}(x)$ is:

$$\boxed{-12}.$$

6. **A** - We are given that the polynomial $R(x)$ has certain remainders when divided by $x-7$ and $x-42$. Specifically, the remainder when $R(x)$ is divided by $x-7$ is 42, and the remainder when $R(x)$ is divided by $x-42$ is 7. We need to find the remainder when $R(x)$ is divided by $(x-7)(x-42)$.

According to the Remainder Theorem, when a polynomial $R(x)$ is divided by $(x-7)(x-42)$, the remainder must be a linear polynomial of the form:

$$R(x) = (x-7)(x-42)Q(x) + ax + b$$

where $ax+b$ is the remainder, and we need to determine a and b.

From the problem, we know:

$$R(7) = 42 \quad \text{and} \quad R(42) = 7$$

Substitute these into the remainder form $R(x) = ax + b$:

$$R(7) = a(7) + b = 42$$

$$7a + b = 42 \quad \text{(Equation 1)}$$

$$R(42) = a(42) + b = 7$$

$$42a + b = 7 \quad \text{(Equation 2)}$$

We now solve the system of linear equations:

$$7a + b = 42 \quad \text{(Equation 1)}$$

$$42a + b = 7 \quad \text{(Equation 2)}$$

Subtract Equation 1 from Equation 2:

$$(42a + b) - (7a + b) = 7 - 42$$

$$35a = -35$$

$$a = -1$$

Substitute $a = -1$ into Equation 1:

$$7(-1) + b = 42$$

$$-7 + b = 42$$

$$b = 49$$

Thus, the remainder when $R(x)$ is divided by $(x-7)(x-42)$ is:

$$\boxed{-x + 49}$$

7. **B**

$$q(x) = x^2 + ax + b$$

$q(x)$ is a factor of both $x^4 + 5x^2 + 9$ and $2x^4 + 6x^2 - 4x + 6$.

$$x^4 + 5x^2 + 9 = (x^2 + ax + b)(x^2 + cx + d)$$

Expanding:

$$x^4 + (a+c)x^3 + (ac+b+d)x^2 + (ad+bc)x + bd$$

$$a + c = 0, \quad ad + bc = 0, \quad ac + b + d = 5, \quad bd = 9$$

$$c = -a, \quad d = b$$

$$-a^2 + 2b = 5, \quad b^2 = 9 \implies b = \pm 3$$

$$\text{If } b = 3:$$

$$-a^2+2(3)=7 \implies -a^2+6=7 \implies -a^2=1 \implies a^2=1 \implies a=\pm 1$$

$$\text{For } a=1, \quad c=-1, \quad d=3$$

$$q(x)=x^2+x+3$$

$$q(2)=2^2+2+3=9$$

$$\boxed{9}$$

8. **C** - Solve each piece for $h(x)=7$:

$$\text{Case 1: } x^3+1=7 \text{ for } x<0$$

$$x^3=6 \implies x=\sqrt[3]{6}$$

(this solution doesn't work since it violates the domain condition).

$$\text{Case 2: } \frac{3}{x}=7 \text{ for } 0<x\leq 4$$

$$x=\frac{3}{7}$$

$$\text{Case 3: } 6x-8=7 \text{ for } x>4$$

$$6x=15 \implies x=\frac{15}{6}=2.5$$

(this solution doesn't work since it violates the domain condition).

$$\boxed{\frac{3}{7}}$$

9. **E** - We are given that x^2+2x+3 is a factor of cx^3+dx^2+4.

$$\text{Let } cx^3+dx^2+4=(x^2+2x+3)(Ax+B).$$

Expand:

$$(x^2+2x+3)(Ax+B)=Ax^3+(2A+B)x^2+(3A+2B)x+3B$$

Compare coefficients:

$$Ax^3+(2A+B)x^2+(3A+2B)x+3B=cx^3+dx^2+0x+4$$

From x^3 and constants: $A=c, \quad 3B=4 \implies B=\frac{4}{3}$Since B must be an integer, no valid solution exists.

10. **D** - We are given that a line passes through $(0,4)$ and intersects the parabola $y=x^2+1$, with the positive difference in x -coordinates being 6.

The equation of the line is $y=mx+4$, where m is the slope. Setting this equal to the parabola's equation:

$$mx+4=x^2+1$$

From the quadratic equation we get

$$6=\sqrt{m^2+12}\to m=2\sqrt{6}\to x=\sqrt{6}\pm 3\to y_2-y_1=2\sqrt{6}(\sqrt{6}+3)+4-2\sqrt{6}(\sqrt{6}-3)-4=\boxed{12\sqrt{6}}$$

11. **C** -

$$|x-2| = 4 - |x|$$
$$x = -1, \quad x = 3$$
$$\text{From } x = -1 \text{ to } x = 0: \quad \text{Area} = \frac{1 \times 2}{2} = 1$$
$$\text{From } x = 0 \text{ to } x = 2: \quad \text{Area} = 2 \times 2 = 4$$
$$\text{From } x = 2 \text{ to } x = 3: \quad \text{Area} = \frac{1 \times 2}{2} = 1$$
$$\text{Total Area} = 1 + 4 + 1 = 6$$
$$\boxed{6}$$

12. **D** - Given:

$$\log_a(b) = 3 \implies b = a^3$$
$$\log_{3a}(3b) = 4 \implies (3a)^4 = 3b$$

Substitute $b = a^3$:

$$81a^4 = 3a^3 \implies 27a = 1 \implies a = \frac{1}{27}$$

Thus:

$$b = \left(\frac{1}{27}\right)^3 = \frac{1}{19683}$$

Compute $\log_{9a}(9b)$:

$$9a = \frac{9}{27} = \frac{1}{3}, \quad 9b = \frac{9}{19683} = \frac{1}{2187}$$
$$\log_{9a}(9b) = \log_{\frac{1}{3}}\left(\frac{1}{2187}\right) = \log_{\frac{1}{3}}\left(\left(\frac{1}{3}\right)^7\right) = 7$$

Answer:

$$\boxed{7}$$

13. **B** - For something to be a function, an arbitrary $f(x)$ can only equal one distinct value. Therefore, looking at the ordered pairs, the only choice that matches this is $\boxed{\textbf{B}}$

14. **C** - The Pythagorean triples that satisfies are 16 and 63 or 33 and 56, in either order. So the answer is $\boxed{4}$

15. **A** -

$$g(x) = \begin{vmatrix} -x & 1 & x \\ x+2 & 2x & 1 \\ 4 & 3 & \frac{2}{x} \end{vmatrix}$$

Expanding the determinant:

$$g(x) = -x \cdot \begin{vmatrix} 2x & 1 \\ 3 & \frac{2}{x} \end{vmatrix} - 1 \cdot \begin{vmatrix} x+2 & 1 \\ 4 & \frac{2}{x} \end{vmatrix} + x \cdot \begin{vmatrix} x+2 & 2x \\ 4 & 3 \end{vmatrix}$$

Calculating each minor:

$$\begin{vmatrix} 2x & 1 \\ 3 & \frac{2}{x} \end{vmatrix} = 2x \cdot \frac{2}{x} - 1 \cdot 3 = 4 - 3 = 1$$

$$\begin{vmatrix} x+2 & 1 \\ 4 & \frac{2}{x} \end{vmatrix} = (x+2) \cdot \frac{2}{x} - 1 \cdot 4 = \frac{2x+4}{x} - 4 = \frac{2x+4-4x}{x} = \frac{-2x+4}{x}$$

$$\begin{vmatrix} x+2 & 2x \\ 4 & 3 \end{vmatrix} = (x+2)\cdot 3 - 2x \cdot 4 = 3x+6-8x = -5x+6$$

Substituting back into $g(x)$:

$$g(x) = -x(1) - 1\left(\frac{-2x+4}{x}\right) + x(-5x+6) = -x + \frac{2x-4}{x} - 5x^2 + 6x$$

Simplifying:

$$g(x) = -x + \frac{2x-4}{x} - 5x^2 + 6x = 5x + \frac{2x-4}{x} - 5x^2$$

Combining over a common denominator:

$$g(x) = \frac{5x^2+2x-4-5x^3}{x} = \frac{-5x^3+5x^2+2x-4}{x}$$

Setting $g(x) = 0$:

$$\frac{-5x^3+5x^2+2x-4}{x} = 0 \quad \Rightarrow \quad -5x^3+5x^2+2x-4 = 0$$

Multiplying by -1:

$$5x^3 - 5x^2 - 2x + 4 = 0$$

For a cubic equation $ax^3 + bx^2 + cx + d = 0$, the sum of the roots is:

$$-\frac{b}{a} = -\frac{-5}{5} = \boxed{1}$$

16. C -

$$x^2 - 7x + 10 = (x-2)(x-5)$$

$$2(2)^3 - 41(2)^2 + 125(2) - 102 = 0 \implies x = 2 \text{ is a shared root}$$

$$f(x) = \left|2x^3 - 41x^2 + 125x - 102\right|^{x^2-7x+10} = 1$$

Solutions occur when:

1. $x^2 - 7x + 10 = 0 \implies x = 5$ (since $x = 2$ leads to 0^0, which is undefined).
2. $\left|2x^3 - 41x^2 + 125x - 102\right| = 1$, resulting in:

$$2x^3 - 41x^2 + 125x - 103 = 0 \quad \text{and} \quad 2x^3 - 41x^2 + 125x - 101 = 0$$

The sum of the roots for each cubic equation is $\frac{41}{2}$.
Total sum of all roots:

$$5 + \frac{41}{2} + \frac{41}{2} = \boxed{46}$$

17. C -

$$f(x) = \frac{x}{x+1}$$

To verify that f is a bijection from $(0, \infty)$ to $(0, 1)$:
Injective:

$$\frac{x_1}{x_1+1} = \frac{x_2}{x_2+1} \implies x_1(x_2+1) = x_2(x_1+1) \implies x_1 = x_2$$

Surjective: For any $y \in (0, 1)$, solve $y = \frac{x}{x+1}$:

$$x = \frac{y}{1-y} \in (0, \infty)$$

Thus, $f(x) = \frac{x}{x+1}$ is both injective and surjective, implying a bijection.

Answer: $\boxed{f(x) = \frac{x}{x+1}}$

18. A -

$$\ln\left(\sqrt{x^2-4x+3}\right) \text{ is undefined when } \sqrt{x^2-4x+3} \leq 0$$

$$x^2 - 4x + 3 \leq 0$$

$$(x-1)(x-3) \leq 0$$

$$1 \leq x \leq 3$$

The integers in this interval are $1, 2, 3$.

$$1 + 2 + 3 = \boxed{6}$$

19. D -

$$2x + y = 20$$

Applying the AM-GM inequality to $2x$ and y:

$$\frac{2x+y}{2} \geq \sqrt{2x \cdot y}$$

$$\frac{20}{2} \geq \sqrt{2xy}$$

$$10 \geq \sqrt{2xy}$$

$$100 \geq 2xy$$

$$xy \leq 50$$

Equality holds when $2x = y$. Substituting into the perimeter equation:

$$2x + 2x = 20 \implies 4x = 20 \implies x = 5$$

$$y = 2x = 10$$

Thus, the maximum area is:

$$A = xy = 5 \times 10 = \boxed{50}$$

20. **B** - We are given three points: $(1, 3)$, $(4, 7)$, and $(5, 11)$ on the quadratic function $g(x) = ax^2 + bx + c$. We want to find $g(2)$.

Substitute the points into the quadratic equation:

1. From $g(1) = 3$:

$$a + b + c = 3$$

2. From $g(4) = 7$:

$$16a + 4b + c = 7$$

3. From $g(5) = 11$:

$$25a + 5b + c = 11$$

Subtract the first equation from the second and third:

$$15a + 3b = 4 \quad (1)$$

$$24a + 4b = 8 \quad (2)$$

Solve these two: 1. Divide (1) by 3: $5a + b = \frac{4}{3}$ 2. Divide (2) by 4: $6a + b = 2$

Subtract to get $a = \frac{2}{3}$. Substitute a back to find $b = -2$, and use these values to find $c = \frac{13}{3}$.

Thus, $g(x) = \frac{2}{3}x^2 - 2x + \frac{13}{3}$.

Finally, calculate $g(2)$:

$$g(2) = \frac{2}{3}(4) - 4 + \frac{13}{3} = 3$$

So, $g(2) = \boxed{3}$.

21. **B** - Factoring and simplifying this equation gives us that the denominator is simply equal to $(x - 4)$. So the only time it is undefined is when $\boxed{x = 4}$

22. **A** - We need the x^2y term from $f(x, y) = (2x - 5y)^3$.

Using the binomial expansion:

$$(2x - 5y)^3 = \sum_{k=0}^{3} \binom{3}{k} (2x)^{3-k}(-5y)^k$$

We want $x^{3-k} = x^2$ and $y^k = y$, so $3 - k = 2$ and $k = 1$.

Substituting $k = 1$:

$$\binom{3}{1}(2x)^{3-1}(-5y)^1 = \binom{3}{1}(2x)^2(-5y)$$

Simplifying:

$$\binom{3}{1} = 3, \quad (2x)^2 = 4x^2, \quad (-5y) = -5y$$

So the term is:

$$3 \times 4x^2 \times (-5y) = -60x^2y$$

The x^2y term is $\boxed{-60x^2y}$.

23. **C** - We check if there's a bijection between $\mathbb{N}$ and each set:

1. $\mathbb{N}$ is countably infinite since the identity function $f(n) = n$ is a bijection from $\mathbb{N}$ to $\mathbb{N}$.

2. $\mathbb{Q}$ is countably infinite. There exists a bijection between $\mathbb{N}$ and $\mathbb{Q}$ by enumerating the rational numbers, so $\mathbb{Q}$ is countable.

3. $\mathbb{R}$ is uncountable. Cantor's diagonal argument shows no bijection between $\mathbb{N}$ and $\mathbb{R}$.

4. $\mathbb{Z}$ is countably infinite, as we can list integers in a sequence that includes all positive and negative integers, so a bijection with $\mathbb{N}$ exists.

Three sets $(\mathbb{N}, \mathbb{Q}, \mathbb{Z})$ are countably infinite.

Thus, $\boxed{3}$ sets are countably infinite.

24. **B** - We need $2 - \sqrt{2x^2 + 4x} \geq 0$ for the function $g(x) = \sqrt{2 - \sqrt{2x^2 + 4x}}$ to be defined.

1. Start by solving $2 - \sqrt{2x^2 + 4x} \geq 0$, which gives $\sqrt{2x^2 + 4x} \leq 2$. Squaring both sides:

$$2x^2 + 4x \leq 4 \quad \Rightarrow \quad x^2 + 2x - 2 \leq 0.$$

The roots are $x = -1 \pm \sqrt{3}$, so $-1 - \sqrt{3} \leq x \leq -1 + \sqrt{3}$.

2. For the inner square root, $2x(x + 2) \geq 0$, the solution is $x \leq -2$ or $x \geq 0$.

Thus the domain is $\boxed{[-1 - \sqrt{3}, -2] \cup [0, \sqrt{3} - 1]}$

25. **A** - To find the range of $g(x) = \sqrt{2 - \sqrt{2x^2 + 4x}}$:

1. The inner expression $\sqrt{2x^2 + 4x}$ ranges from 0 to 2 since $0 \leq \sqrt{2x^2 + 4x} \leq 2$.

2. Therefore, $2 - \sqrt{2x^2 + 4x}$ ranges from 0 to 2.

3. Taking the square root of this gives $g(x) = \sqrt{2 - \sqrt{2x^2 + 4x}}$, which ranges from 0 to $\sqrt{2}$.

Thus, the range of $g(x)$ is $[0, \sqrt{2}]$.

26. **D** - We are given $g(x) = x^2 + \frac{1}{x^2} + 5$ and need to find its minimum value for $x \neq 0$.

By the AM-GM inequality, we know that for $a = x^2$ and $b = \frac{1}{x^2}$, we have:

$$a + b \geq 2\sqrt{ab}.$$

Since $ab = x^2 \cdot \frac{1}{x^2} = 1$, we get:

$$x^2 + \frac{1}{x^2} \geq 2.$$

Thus, $g(x) = x^2 + \frac{1}{x^2} + 5 \geq 2 + 5 = 7$.

The minimum value of $g(x)$ is $\boxed{7}$, which occurs when $x^2 = 1$.

27. **C** - We use shoelace formula:

$$\begin{aligned} \text{Area} &= \frac{1}{2}\left|0\cdot 3+2\cdot 25+5\cdot 15+2\cdot 1-(2\cdot 1+5\cdot 3+2\cdot 25+0\cdot 15)\right| \\ &= \frac{1}{2}\left|0+50+75+2-(2+15+50+0)\right| \\ &= \frac{1}{2}\left|75-15\right| = \frac{1}{2}\times 60 = \boxed{30}. \end{aligned}$$

28. **A** - We can see $f(x) = a^x$. Then, plugging in our conditions, we see that $f(x) = 2^x$. So $f(2025) = \boxed{2^{2025}}$

29. **B** - Vietas tells us the sum is $\boxed{\frac{4}{3}}$

30. **B** - $f(x+2025) = x^2+2x-5 \rightarrow f(x) = (x-2025)^2+2(x-2025)-5$ So the product of the roots is $2025^2-2(2025)-5 = (2025)(2023)-5 = \boxed{4096570}$