2021 Log1 Contest: THETA answers

1. **D**, the converse of the statement is switching the two clauses, the inverse of the statement is negating both clauses, and the contrapositive of the statement. Thus, converting the statement one step at a time we get:

If you get this question correct, then you will feel good about yourself. (original)

If you do not get this question correct, then you will not feel good about yourself. (inverse)

If you feel good about yourself, then you will feel get this question correct. (contrapositive)

If you get this question correct, then you will feel good about yourself. (converse)

If you do not feel good about yourself, then you will not get this question correct. (contrapositive)

If you feel good about yourself, then you will get this question correct. (inverse)

If you get this question correct, then you will feel good about yourself. (converse)

Thus, we are left with our original statement. The easiest way to visual this kind of problem is to realize that taking the inverse of the inverse is the same as the original statement (this is true for the converse and contrapositive as well). Thus, the pairs cancel with each other out leaving us with the original statement.

2. **A,** since $Q(x)$ is an even function we know that $Q(x) = Q(-x)$. Thus, $Q(-2) = -6$. Furthermore, we know that an even parabolic function can be written in the form $ax2 + bx + c$. We know that our leading coefficient is 1. We are not left with $Q(x)$. $= x2 + bx + c$ Plugging in 2 and -2 respectively:

$22 + 2b + c = -6$

$(-2)2 + (-2)b + c = -6$

We now have 2 equations with two unknowns. Using elimination:

$4 + 2b + c = -6$

$4 - 2b + c = -6$

$8 + 2c = -12$

$2c = -20$

$c = -10$ which is our y-intercept as the c value determines the parabolas vertical shift as our parabola is centered around the y-axis.

3.

A, since n = $\sqrt{x-\sqrt{x-\sqrt{x-\sqrt{x-\cdots}}}}$ we can substitute n for the first time where the sequence repeats. Thus,

$$n = \sqrt{x - n}$$

$$n^2 = x - n$$

$$n^2 + n = x$$

4. **A**, firstly we must use the area of a sector formula to solve for the radii of Circle A and Circle B respectively. Area of Sector = $\frac{1}{2}r^2\theta$ when θ is in radians Thus,

$12 = (\frac{1}{2})(r_a^2)(\frac{\pi}{5})$

$12 = (\frac{\pi}{10})(r_a^2)$

$$(r_a^2) = \frac{120}{\pi}$$

$$r_a = \sqrt{\frac{120}{\pi}}$$

$$14 = (\frac{1}{2})(r_b^2)(\frac{7\pi}{11})$$

(continued on next page)

(#4 answer continued)

14 = $(\frac{7\pi}{22})(r_b^2)$

$$(r_b^2) = \frac{44}{\pi}$$

$$r_b = \sqrt{\frac{44}{\pi}}$$

Now that we have the radii of Circle A and Circle B we can find the ratio between the volume of Sphere B to Sphere A. The volume of a sphere is = $\frac{4}{3}\pi r^3$. Thus, the ratio of the volume between our spheres is

Ratio = $\frac{\frac{4}{3}\pi r_b^3}{\frac{4}{3}\pi r_a^3}$

Ratio = $\frac{r_b^3}{r_a^3}$

Ratio = $\frac{(\frac{44}{\pi})(\sqrt{\frac{44}{\pi}})}{(\frac{120}{\pi})(\sqrt{\frac{120}{\pi}})}$

Ratio = $\frac{44\sqrt{44}}{120\sqrt{120}}$

Ratio = $\frac{88\sqrt{11}}{240\sqrt{30}}$

Ratio = $\frac{11\sqrt{11}}{30\sqrt{30}}$

Ratio = $\frac{11\sqrt{330}}{900}$

5. **C**, firstly we must realize a couple of things. The diameter of the circle cuts through 2 vertices of the hexagon. The hexagon is composed of 6 identical equilateral triangles whose each side is equal to the radius of the circle since the radius connects the vertex of the hexagon to its center. Thus,

$$64\pi = \pi r^2$$

$$64 = r^2$$

$$r = 8$$

$$Area\ Equilateral\ Triangle = \frac{s^2\sqrt{3}}{4}$$

$$Area\ Equilateral\ Triangle = \frac{8^2\sqrt{3}}{4}$$

$Area\ Equilateral\ Triangle = 16\sqrt{3}$ however, we multiply this by 6 as there are 6 equilateral triangles that compose a regular hexagon. Thus, we get $96\sqrt{3}$.

6. **D**, suppose that we call the roots of $P(x)$ x, y, and z. The product of the roots taken two at a time is

equal to $(xy)(yz)(xz) = (xyz)^2$. Thus, we are simply trying to find the product of the roots, squared. This can be found by Vieta's formula. The product of the roots is = $\frac{-constant}{a}$ if the degree of the highest term is odd and $\frac{constant}{a}$ if the degree of the highest term is even. In our case,

Product = $\frac{-(-16)}{1} = 16$

$$16^2 = 256$$

7) **C and E are acceptable.**

I is true because if we have 2 corresponding angles then the third must also be a corresponding angle. With all three congruent angles we can determine that one triangle is simply a dilation of another as the sides are constructed from the angles.

II is false because if we have 2 corresponding sides we must have the included angle (the angle that lies at the vertex of the two sides) as opposed to any two angles. The included angle with the two sides proves congruency

III is true because if we have 3 corresponding and proportional sides between the triangles we know (like stated in I) that the triangles are simply a dilation of each other.

Thus, only I and III are true. **= Answer C**

Statement I says "If two angles of a triangle are congruent". It should say "If two corresponding angles of two triangles are congruent". Thus Statement I is false and only Statement III is true. **= Answer E**

8. **A**, to find the center of the conic we need to put it into its standard form and to do this we need to complete the square. Thus,

$$2x^2 - y^2 - 16x + 10y - 41 = 0$$

$$2(x^2 - 8x) - (y^2 - 10y) = 41$$

$$2(x^2 - 8x + 16) - (y^2 - 10y + 25) = 41 + (2)(16) + (-1)(25)$$

$$2(x - 4)^2 - (y - 5)^2 = 48$$

$$\frac{(x - 4)^2}{24} - \frac{(y - 5)^2}{48} = 1$$

Thus, the center occurs at (4,5). 4 + 5 = 9

9. **B**, since matrix addition and subtraction is commutative we can find the resulting matrix by adding and subtracting with each number in its respective position. Thus,

$\begin{bmatrix} 6 + 7 - 5 & 4 + 4 - 9 \\ -2 + 2 + -3 & -1 + -5 + 8 \end{bmatrix} = \begin{bmatrix} 8 & -1 \\ -3 & 2 \end{bmatrix}$ the sum of the elements of this matrix is

8 + -1 + -3 + 2 = 6. Therefore, the units digit is 6.

10. **A**, first begin with writing only the units digits of 2 in increasing powers and then 3 in increasing powers. These numbers are 2,4,8,6,2,4,8,2… and 3,9,7,1,3,9,7,1… respectively. As we see every 4th power our sequence repeats and from this pattern we only need to concentrate on the remainder of the nth power to determine the value of its units digit. Luckily, the divisibility rule for 4 is to divide only the last two digits. Thus,

$2^{61203478} = 2^{78}$ which leaves a remainder of 2 which correlates to the 2nd term of our sequence which is 4.

$3^{71239563} = 3^{63}$ which leaves a remainder of 3 which correlates to the 3rd term of our sequence which is 7.

4 + 7 = 11. Thus, our units digit is 1.

11. **C**, to construct an orthocenter the intersection of the altitudes of each side is used, to construct a centroid the intersection of the medians of each side is used, to construct an incenter the intersection of the angle bisectors of each side is used, to construct a circumcenter the intersection of the perpendicular bisectors is used.

12. **E**,

3 + (5 − 2)2 − 3 × 2 + 6 − 3 ÷ (1 + 2) − 1

3 + (3)2 − 3 × 2 + 6 − 3 ÷ (1 + 2) − 1

3 + 9 − 3 × 2 + 6 − 3 ÷ (1 + 2) − 1

3 + 9 − 3 × 2 + 6 − 3 ÷ (3) − 1

3 + 9 − 6 + 6 − 1 − 1

12 − 6 + 6 − 1 − 1

10

13.

B This infinite sum is geometric. Observe that by change of base formula, we can phrase each term in log base 3, because $\log_9(x) = \frac{\log_3(x)}{\log_3(9)}$ and $\log_{81}(x) = \frac{\log_3(x)}{\log_3(81)}$, so there is a common ratio of ½. By infinite sum of a geometric series, the sum is $2\log_3(x)$, which has a solution when $x = 3^9$.

14. This is a Caesar cipher with a shift of 13, also known as ROT13.

Never gonna give you up, never gonna let you down, never gonna run around and desert you.

15. This is a keyword cipher using the keyword ''PRASEODYMIUM.'' A keyword cipher works similarly to a cryptogram except that a keyword appears at the leftmost part of the decryption table, as follows.

Ciphertext: ABCDEFGHIJKLMNOPQRSTUVWXYZ

Plaintext: PRASEODYMIUBCFGHJKLNQTVWXZ

I am serious, and don't call me Shirley.