This is the Theta 3D Geometry test. E. NOTA specifies None of the Above. The questions are arranged roughly in increasing difficulty. That being said, you may wish to take a look at number 30 even if you don't get there. Good luck!

1. Find the volume of a cube with side length 2.

   A. $\sqrt[3]{2}$ B. 4 $\boxed{\text{C. } 8}$ D. 16 E. NOTA

   Solution: $2^3 = 8$.

2. Find the volume of a sphere inscribed in a cube with side length 2.

   $\boxed{\text{A. } \frac{4\pi}{3}}$ B. $\frac{32\pi}{3}$ C. $\frac{4\pi}{9}$ D. $\frac{32\pi}{9}$ E. NOTA

   Solution: The sphere has radius 1.

3. Find the volume of a cube inscribed in a sphere inscribed in a cube with side length 2.

   A. $\frac{8}{27}$ B. $\frac{8\sqrt{3}}{27}$ C. $\frac{8}{9}$ $\boxed{\text{D. } \frac{8\sqrt{3}}{9}}$ E. NOTA

   Solution: The cube has a space diagonal of length 2, and so a size length of $\frac{2}{\sqrt{3}}$. The volume is thus $\frac{8}{3\sqrt{3}} = \frac{8\sqrt{3}}{9}$.

4. Cone P has one-third the radius and three times the height of cone Q. Find the ratio of the volume of cone Q to cone P.

   A. $1:27$ B. $1:3$ $\boxed{\text{C. } 3:1}$ D. $27:1$ E. NOTA

   Solution: The ratio of the radii must be squared, while the ratio of the heights remains the same. $\frac{3^2}{3} = 3:1$.

5. The centers of three spheres of radius $\sqrt{3}$ form an equilateral triangle with side length 6. Find the radius of the smallest sphere which can enclose these three spheres.

   A. $6+\sqrt{3}$ B. $2\sqrt{3}$ $\boxed{\text{C. } 3\sqrt{3}}$ D. $6\sqrt{3}-3$ E. NOTA

   Solution: We need to find the farthest distance on the sphere from the center of the three spheres. The altitude of the triangle is $3\sqrt{3}$, and so the distance from the center to the center of the spheres is $\frac{2}{3} \cdot 3\sqrt{3} = 2\sqrt{3}$. Then we just need to the radius of the sphere for a total of $3\sqrt{3}$.

6. DZ the fly is inside a cube of side length 4, and tied to a corner of the cube by a rope of length 3. How much volume does DZ have to fly around in?

   $\boxed{\text{A. } \frac{9\pi}{2}}$ B. 36π C. $\frac{63\pi}{2}$ D. $\frac{27\pi}{2}$ E. NOTA

   Solution: The area he can fly is a sphere because he is tethered by a rope. However, since he is inside a cube, he can only fly in one octant of the sphere. $\frac{1}{8} \cdot \frac{4\pi}{3} 3^3 = \frac{9\pi}{2}$.

7. Two spheres with centers A and B have radius 1. If the distance form A to B is also 1, find the area of the circle whose circumference is formed by the intersections of the two spheres.

A. π | $\boxed{\text{B. } \frac{3\pi}{4}}$ | C. $\frac{\pi}{2}$ | D. $\frac{\pi}{4}$ | E. NOTA

Solution: The distance from the intersection of the spheres to the common radius of the spheres is the radius of the circle we seek. Slicing the figure along this radius reveals an equilateral triangle of side length 1, where the base is the common radius and the other two sides are other radii of each sphere. Then the altitude of the triangle is $\frac{\sqrt{3}}{2}$, and so the area of the circle with that radius is $\frac{3\pi}{4}$.

8. A cube of side length 6 has all of its vertices truncated (sliced off) by planes such that the shortest distance from the center of the cube to each face created by the truncation is 3. Let V_n represent the polyhedron formed by n iterations of this process. Find

$$\lim_{n\to\infty} V_n$$

A. 27 | B. 72 | C. 27π | D. 32π | $\boxed{\text{E. NOTA}}$

Solution: The limit will be when there are no vertices on the polyhedron which are farther than 3 from the center. This is the definition of a sphere with radius 3, which has volume 36π.

9. Find the volume of a regular tetrahedron of side length 6.

$\boxed{\text{A. } 18\sqrt{2}}$ | B. $18\sqrt{3}$ | C. $24\sqrt{2}$ | D. $24\sqrt{3}$ | E. NOTA

Solution: The volume of a tetrahedron of side length s is $\frac{s^3\sqrt{2}}{12}$. If this is not known, it can be derived through some cleverly chosen cross sections.

10. Find the radius of a sphere inscribed in a right circular cone with radius 3 and height 4.

A. 1 | $\boxed{\text{B. } \frac{3}{2}}$ | C. $\frac{2}{3}$ | D. $\frac{4}{3}$ | E. NOTA

Solution: Taking a cross-section along the axis of the cone reveals a circle inscribed in a triangle. Drawing a radius to the bottom and side of the triangle, and drawing the altitude of the triangle, reveals a set of similar right triangles, one with its side on the base of the triangle, and one with its side as the drawn radius of the circle. Because they are right and share an angle, they are similar. We can set up a ratio, then, as $\frac{3}{5} = \frac{r}{4-r}$, where this is the ratio of the short side to the hypotenuse. Solving this yields $r = \frac{3}{2}$.

11. Find the number of edges in a convex polyhedron with 11 faces and 33 vertices.

A. 20 | B. 46 | $\boxed{\text{C. } 42}$ | D. 24 | E. NOTA

Solution: Euler's formula states that $F + V = E + 2$. Substitution yields $11 + 33 - 2 = 42$.

12. What is the intersection of the graphs $z^2 = x^2 + y^2$ and $x = 2$?

    A. Circle   B. Ellipse   C. Parabola $\boxed{\text{D. Hyperbola}}$ E. NOTA

    Solution: The graph is of a double mapped cone (to see this, consider each value of z is a circle with radius $|z|$). Then intersecting with $x = 2$ is the definition of a hyperbola.

13. What is the maximum number of spheres that can be put into a configuration such that each sphere is externally tangent to every other sphere?

    A. 3   B. 4 $\boxed{\text{C. 5}}$ D. 6   E. NOTA

    Solution: The first impression is four, organized in a stack like a tetrahedron. However, one additional sphere can be nested in the center of these four spheres, for a total of five.

For the next three questions, consider cube $ANDYA'N'D'Y'$ which has base $ANDY$, A adjacent to A', N adjacent to N', D adjacent to D', Y adjacent to Y', and $AN = 2$.

14. Find the area of triangle ADY'.

    A. $\sqrt{2}$ B. $\sqrt{3}$ C. $2\sqrt{2}$ $\boxed{\text{D. } 2\sqrt{3}}$ E. NOTA

    Solution: The triangle is equilateral with side length $2\sqrt{2}$, so has area $2\sqrt{3}$

15. Let P be the midpoint of AA' and Q be the midpoint of DD'. Find the area of quadrilateral $PNQY'$.

    A. $\sqrt{3}$ B. $\sqrt{6}$ C. $2\sqrt{3}$ $\boxed{\text{D. } 2\sqrt{6}}$ E. NOTA

    Solution: The figure is a rhombus with diagonals of length $PQ = 2\sqrt{2}$ and $NY' = 2\sqrt{3}$, so the area is $\frac{1}{2} \cdot 2\sqrt{2} \cdot 2\sqrt{3} = 2\sqrt{6}$.

16. The plane that bisects and is perpendicular to diagonal AD' intersects the edges of the cube at six different points. Find the area of the convex polygon formed by these six points.

    A. $2\sqrt{3}$ B. $2\sqrt{6}$ $\boxed{\text{C. } 3\sqrt{3}}$ D. $3\sqrt{6}$ E. NOTA

    Solution: The shape is a regular hexagon with side length $\sqrt{2}$, so the area is $3\sqrt{3}$.

A right cylinder has radius 2 and height 100. Plane P makes a 30° angle with the plane containing one of the cylinder's faces, and intersects the center axis of the cylinder at a point 6 units from that base, cutting the cylinder into two sections. For the next two questions, consider the smaller of those sections.

17. Find the volume of this figure.

A. 12π $\quad$ $\boxed{\text{B. } 24\pi}$ $\quad$ C. 36π $\quad$ D. 48π $\quad$ E. NOTA

Solution: A neat trick is to take two of the figures and stick them together to form a whole cylinder with height 12 and radius 2. Then half of this volume is $\frac{1}{2} \cdot 48\pi = 24\pi$.

18. Find the surface area of this figure.

A. $\frac{4\pi\sqrt{3}}{3} + 24\pi$ $\quad$ B. $\frac{4\pi\sqrt{3}}{3} + 28\pi$ $\quad$ $\boxed{\text{C. } \frac{8\pi\sqrt{3}}{3} + 28\pi}$ $\quad$ D. $\frac{8\pi\sqrt{3}}{3} + 24\pi$ $\quad$ E. NOTA

Solution: To find the lateral surface area, the same trick as the previous question can be used. This results in $\frac{1}{2} \cdot 48\pi = 24\pi$ (where the 48π is the lateral surface area of a cone with height 12 and radius 2). The circular base is quickly 4π. That leaves the elliptical base. The minor axis of the ellipse is clearly 2 as it coincides with the radius of the cylinder. To find the major axis, we consider the end right triangle in the cross section along this axis and the axis of the cylinder. This has a base of 4, and it is a 30-60-90 right triangle, so the hypotenuse is $\frac{8\sqrt{3}}{3}$. This is twice the major axis of the ellipse. Thus, the area of the ellipse is $\pi \cdot \frac{4\sqrt{3}}{3} \cdot 2 = \frac{8\pi\sqrt{3}}{3}$. Summing the lateral, elliptical and circular areas gives $\frac{8\pi\sqrt{3}}{3} + 28\pi$.

A cube has side length 3. On each open face of this cube, a cube with $\frac{1}{3}$ the side length is attached to the center. This process is repeated for each new cube, and so on forever. (For reference, there are 7, 37, then 187 total cubes after the first 1, 2 and 3 iterations). For the next two questions, consider this fractal.

19. Find the volume of this figure.

A. $\frac{729}{22}$ $\quad$ $\boxed{\text{B. } \frac{378}{11}}$ $\quad$ C. $\frac{243}{7}$ $\quad$ D. $\frac{81}{2}$ $\quad$ E. NOTA

Solution: At the ith iteration of the process, we add $6 \cdot 5^{i-1}$ cubes (5^{i-1} for each side of the original cube, where this is because each new addition leaves five new faces for the next), each with side length $\frac{1}{3^{i-1}}$. Then the total volume added at the ith iteration is $6 \cdot 5^{i-1} \cdot \frac{1}{3^{3i-3}} = 6 \cdot \frac{5^{i-1}}{27^{i-1}}$. To calculate the total volume, we sum this for iterations from $i = 1$ to infinity (it is an infinite geometric series; the formula is $\frac{a_1}{1-r}$) to get $\frac{81}{11}$, and then add the original cube, for a total of $\frac{378}{11}$.

20. Find the surface area of this figure.

A. 99 $\quad$ $\boxed{\text{B. } 108}$ $\quad$ C. $\frac{441}{4}$ $\quad$ D. $\frac{243}{2}$ $\quad$ E. NOTA

Solution: Using similar reasoning to the previous problem, the total area added on the ith iteration is $6 \cdot 5^{i-1} \cdot \left(4 \cdot \frac{1}{9^{i-1}}\right)$. This surface area added per cube has a factor of 4 because there are six faces per cube, but one is covered up, and the addition of the cube coveres up another equal area on the previous cube. We can then sum this from $i = 1$ to infinity and add the original 54 surface area to get 108.

For the next two questions, consider the plane which passes through the points $(3, 0, 0)$, $(0, 4, 0)$, and $(0, 0, 4)$.

21. Find the volume of the tetrahedral region bounded the first octant (positive x, y, z) and the plane.

    A. 6 $\boxed{\text{B. } 8}$ C. 10   D. 12   E. NOTA

    Solution: The figure is a pyramid. The base has area $3 \cdot 4 \cdot \frac{1}{2} = 6$, and the height is 4, for a volume of 8.

22. Find the distance from the origin to the plane.

    A. $\frac{5\sqrt{34}}{17}$ $\boxed{\text{B. } \frac{6\sqrt{34}}{17}}$ C. $\frac{8\sqrt{34}}{17}$ D. $\frac{12\sqrt{34}}{17}$ E. NOTA

    Solution: We can treat the figure as a pyramid with a base on the plane. Then, the height of the pyramid is the distance we seek. We have calculated the volume as 8 already, so we can must the area of the plane. The side lengths of the base are 5, 5 and $4\sqrt{2}$. Via Heron's or dropping an altitude, the area is $2\sqrt{34}$. Then we have $8 = \frac{1}{3} \cdot 2\sqrt{34} \cdot h$, and we find $h = \frac{6\sqrt{34}}{17}$

23. Find the volume of the figure formed when the region bound in the first quadrant by the equation $y = 4 - |x - 4|$ is revolved around the y-axis.

    A. $\frac{256\pi}{3}$ B. 96π C. $\frac{1024\pi}{9}$ $\boxed{\text{D. } 128\pi}$ E. NOTA

    Solution: Consider the cone formed by rotating the region bounded by the coordinate axes and $y = 8 - x$ over the y-axis. This cone has radius and height 8 and thus volume $\frac{512\pi}{3}$. We now need to subtract the solid formed when the region bounded by the y-axis, $y = 8 - x$, and $y = x$ is rotated over the y-axis. If this region is cut in half by the line $y = 4$, then the solid becomes two cones with radius and height 4. Their combined volume is $2 \cdot \frac{64\pi}{3} = \frac{128\pi}{3}$, and subtracting this from the larger solid gives the volume of the desired solid as 128π.

24. Three mutually tangent spheres of radius 1 are placed on a table. A sphere of radius 2 is placed on top of them so that it is tangent to each smaller sphere. Find the minimum distance between the table and the largest sphere.

    A. $\frac{3}{2}$ B. $\frac{5\sqrt{3}}{3} - 1$ C. $\sqrt{5} - 1$ $\boxed{\text{D. } \frac{\sqrt{69}}{3} - 1}$ E. NOTA

    Solution: The centers of the three smaller spheres form an equilateral triangle with side length 1, so its centroid is a distance of $\frac{2}{\sqrt{3}}$ from each vertex. Connecting radii of the large sphere and a small sphere forms a hypotenuse of length 3, so by the Pythagorean Theorem, the distance from the centroid to the top of the large sphere is $\sqrt{9 - \frac{4}{3}} = \frac{\sqrt{69}}{3}$. Thus, the distance from the centroid to the bottom of the large sphere is $\frac{\sqrt{69}}{3} - 2$ and the distance from the table to the bottom of the large sphere is $\frac{\sqrt{69}}{3} - 1$.

25. A cube has two adjacent vertices at $(4,3,25)$ and $(4,3,27)$. If each of these cube's vertices are transformed by being multiplied by the matrix

$$\begin{bmatrix} 2 & 1 & 3 \\ -2 & -8 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

then what is the volume of the resulting figure?

A. 0 B. 8 $\boxed{\text{C. } 16}$ D. 64 E. NOTA

Solution: The determinant of the matrix is the factor by which volume is scaled under transformation. The determinant here is 22, so the answer is $2^3 * 2 = 16$.

26. Tetrahedron $ABCD$ has side length 6. Sphere X has diameter AB, and sphere Y has diameter CD. Find the area of the circle whose circumference is formed by the intersections of the two spheres.

A. $\frac{27\pi}{4}$ B. $\frac{27\pi}{2}$ C. $\frac{9\pi}{4}$ $\boxed{\text{D. } \frac{9\pi}{2}}$ E. NOTA

Solution: First we need to find the distance between the centers of the spheres. We can take a cross section of the tetrahedron to get an isosceles triangle with legs $3\sqrt{3}$ and base 6, where the altitude is this distance and is equal to $\sqrt{(3\sqrt{3})^2 - 3^2} = 3\sqrt{2}$. Now the problem is similar to problem 7, except the triangle from the cross section is isosceles with legs 3 and base $3\sqrt{2}$. The radius is the altitude of this triangle, which is $\frac{3}{\sqrt{2}}$, so the area is $\frac{9\pi}{2}$.

27. DZ the fly has escaped his cubic prison, but finds himself tied down yet again. His new leash is of length 6, and he is tied to the center of a thin (negligible height) disc of radius 3. Assuming the disc is fixed in place, how much volume does DZ have to fly around?

A. $72\pi + 27\pi^2$ B. $72\pi + 54\pi^2$ $\boxed{\text{C. } 144\pi + 27\pi^2}$ D. $144\pi + 54\pi^2$ E. NOTA

Solution: There are two sections to consider. First, DZ can fly anywhere above the disc, which is a volume equal to a hemisphere with radius 6, or 144π. Second, DZ can fly below the disc. Taking a cross-section normal to the disk shows that this results in two semi-circles below the disk. Thus, the volume DZ can fly below the disc is exactly half of a torus with inner radius 3 and outer radius 3, which is a volume of $\frac{1}{2}(\pi 3^2)(2\pi \cdot 3) = 27\pi^2$. Thus, the total volume DZ can fly in is $144\pi + 27\pi^2$.

28. A sphere with radius 3 has diameter AB. Two planes which contain AB and intersect at an angle of $40°$ cut the sphere into two small and two large wedges. Find the outer surface area of one of the smaller wedges (not including the two semi-circular faces).

$\boxed{\text{A. } 4\pi}$ B. 8π C. 32π D. 64π E. NOTA

Solution: Because they pass through a diameter, the sphere is split into sections in ratios corresponding to the angles. One of the small sections thus has an area of $\frac{40}{360} = \frac{1}{9}$ the total area: $\frac{1}{9} \cdot 4\pi \cdot 3^2 = 4\pi$

29. A sphere with radius 3 has points A, B, and C on its surface. A bug walks along the surface of the sphere directly from A to B, then B to C, then from C back to A. The path the bug traces out is known as a spherical triangle. Given that the angles of this spherical triangle are 60° at A, 150° at B, and 90° at C, find the area of this spherical triangle. (Hint: Think about the previous problem. What is the total area of the wedges the spherical triangle would make? What is overcounted?)

A. 3π B. $3\pi\sqrt{3}$ $\boxed{\text{C. } 6\pi}$ D. 9π E. NOTA

Solution: The hint is extremely useful here. First consider the spherical triangle $A'B'C'$ directly opposite of ABC. If we split the sphere along the sides of the triangle along diameters AA', BB', and CC' like in the previous problem, then when we sum the areas of the six wedges we create, we actually end up covering the entire sphere: $36\pi \cdot (2\frac{60}{360} + 2\frac{150}{360} + 2\frac{90}{360}) = 60\pi$. The only issue is that we have counted both ABC and $A'B'C'$ three times each (one for each vertex). Thus, we subtract the area of the sphere $60\pi - 36\pi = 24\pi$, which leaves 4 extra counts of the area. We then divide by 4 to get 6π. This problem is particularly tricky to reason about and really puts your 3D thinking to the test. If you got it right, or understand it now, congrats!

30. A cube is filled with water such that when its base is parallel to the ground, the water fills $\frac{1}{3}$ of the cube. The cube is then tilted such that its space diagonal is normal to the ground. How much of the cube does the water fill now?

A. $\frac{1}{6}$ $\boxed{\text{B. } \frac{1}{3}}$ C. $\frac{1}{4}$ D. $\frac{1}{2}$ E. NOTA

Solution: No water was added or removed, so it still fills 1/3 of the cube, regardless of the orientation.