2022 MU ALPHA THETA NATIONAL CONVENTION
THETA APPLICATIONS ANSWERS and SOLUTIONS

ANSWERS :

1. A
2. B
3. E (56)
4. D
5. B
6. C
7. D
8. B
9. A
10. A
11. E ($\frac{100}{7}$)
12. A
13. C
14. D
15. D
16. D
17. C
18. A
19. B
20. B
21. B
22. B
23. D
24. C
25. D
26. C
27. A
28. E (27)
29. C
30. B

SOLUTIONS :

1) **A)** $+$
For this question, we can see, after much trial and error, that $\left(5 - \frac{2}{10}\right) \times 5$ gives 24, so the only operation that is not used is $+$.

2) **B)** $3\sqrt{17}$
In order to solve this efficiently, we can use the classic reflection trick. Because the shortest distance between two points is a line, you just need to reflect one of the points over $y = 3x + 6$ and calculate the straight-line distance to the other point. We will reflect (3, 5) because it is already sitting on a nice perpendicular (the line $y = -\frac{1}{3}x + 6$). Reflecting (3, 5) over $y = 3x + 6$ gives us the point (-3, 7). Then, we just use distance formula from (-3, 7) to (9, 10) to get $3\sqrt{17}$.

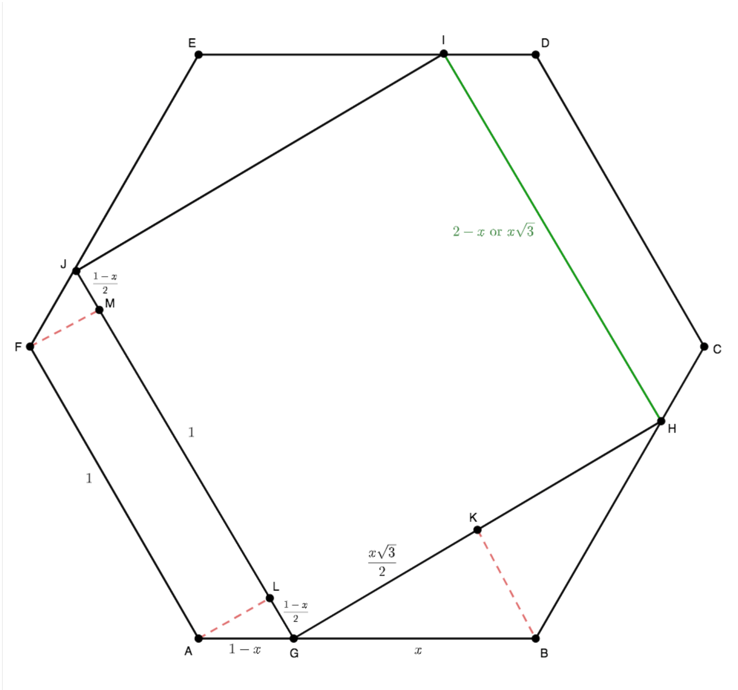
3) **E) 56**
The shortest route consists of 5 movements to the right and 3 movements up. Therefore, we just need to arrange in what order those movements are, so this question is the same as asking: how many ways can you arrange RRRRRUUU. We can use some simple combinatorics and see the answer is $\binom{8}{3}$, or 56.

4) **D) 21**
If we call the three roots $r,\ s,\ t$ and u, we can expand $(r + s + t + u)^2$ to get $r^2 + s^2 + t^2 + u^2 + 2rs + 2rt + 2ru + 2st + 2su + 2tu$, which looks similar to what we are looking for. Then, we can see that $2rs + 2rt + 2ru + 2st + 2su + 2tu$ is just 2 times the sum of the roots 2 at a time. Therefore, the expression we are looking for is just $(sum\ of\ the\ roots)^2 - 2(sum\ of\ the\ roots\ two\ at\ a\ time)$. Now, we can use Vieta's formula to see that the final answer is 21.

5) **B)** $\frac{8\sqrt{3}-12}{3}$

First, let us call the hexagon $ABCDEF$ with side length 1 and the square $GHIJ$ for simplicity (the side length is irrelevant because we are dealing with probabilities). We can now inscribe the square in such a way that G lies on AB, splitting it into AG and GB, and H lies on BC. We will call AG shorter than GB. Now, we can drop an altitude from B intersecting GH at K as well as an altitude from A intersecting GJ at L and from F intersecting GJ at M. We can label GB as x, making AG equal to $1-x$ and GK equal to $\frac{x\sqrt{3}}{2}$. Because AG is $1-x$, we can see that GL is $\frac{1-x}{2}$, and by symmetry, $GL = JM$. Finally, $ML = 1$. So, the side of the square can be represented as both $2(GK) = x\sqrt{3}$, and $1 + 2(GL) = 1 + 1 - x = 2 - x$. If we set these two equal and solve, we get $x = \sqrt{3} - 1$. This makes the side length of the square $3 - \sqrt{3}$ and the area is $12 - 6\sqrt{3}$. And lastly, we see that the area of the hexagon is $\frac{3\sqrt{3}}{2}$. Dividing, we get our final answer of $\frac{8\sqrt{3}-12}{3}$.

Picture for reference:



6) **C)** $\frac{6}{7}$

There are three snakes, so there are 2^3 or 8 possible gender orientations: MMM, MMF, MFM, FMM, MFF, FMF, FFM, FFF. The fact that there was hissing tells us that at least one of the snakes is male, eliminating the FFF case leaving us with 7 possible cases. We know that you will only die if the snakes are MMM, so the other 6 cases mean you survive. So, the answer is $\frac{6}{7}$.

7) **D) 9**
We can reflect the image about AC to create triangle ACD. Now, the image is a circle inscribed inside triangle ABD. We can now use the formula $A = sr$ where A is the area of the triangle, s is the semi-perimeter of the triangle, and r is the radius of the inscribed circle. We can see that the semi-perimeter is $\frac{18+15+15}{2} = 24$ and the area of ABD is $9(12) = 108$. Plugging these into $A = sr$, we can see the radius of the circle is $\frac{9}{2}$, which makes the diameter 9.

8) **B)** $\mathbf{\frac{3}{4}}$
We first need to convert everything into the same base so we can log both sides and get rid of the exponents. Luckily, everything is easily convertible into powers of 2. The equation we get is $2^{x+1} = 2^{2x-2} \cdot 2^{3x}$. We can now simplify the right side and log everything base 2 to get the equation $x + 1 = 5x - 2$. Solving, we get $x = \frac{3}{4}$.

9) **A)** $(\mathbf{8}, \infty)$
For this question, we need to peel back the layers of the stacked logs. First of all, we know that $log_4\left(log_3\left(log_2(x)\right)\right) > 0$. From there, we can raise both sides to the fourth power and see that $log_3\left(log_2(x)\right) > 1$. Next raise both sides to the third power, $log_2(x) > 3$. Finally, we raise both sides to the second power and we see that $x > 8$, or $(8, \infty)$.

10) **A) 29**
For us to count this, we need to break it down into smaller cases. There is obviously only one way to get to the first step. There are two ways of getting to the second step (either you can go there directly from the bottom, or you can take the first step then the second step). There are four ways of getting on the third step: you can go there directly from the bottom, go there from the first step, or go there from the second step. But because the second step has two ways of getting there, we need count that twice. Now, the fourth step can be reached from the first step (1 way), the second step (2 ways) and the third step (4 ways). Therefore, there are seven ways to get to the fourth step. The fifth step is a trap so we can label it as 0 ways, and we can continue adding the values of the previous three steps until we reach the 8th step, to get a final answer of 29.

11) **E)** $\mathbf{\frac{100}{7}}$
This is a classic concentration question, and it can be solved with ratios. We have 20L of 20% magic and 80% health, which means we have 4L of magic and 16L of health inside the cauldron initially. Next, we see that we need the final mixture to be 45% (or $\frac{9}{20}$) magic. Our mixture that we can add is 80% magic and 20% health, which means that for every liter of that mixture that we add, we are adding 0.8L magic. Now, we can set up our equation, which is $\frac{liters\ of\ magic}{liters\ total} = \frac{4+0.8x}{20+x} = \frac{9}{20}$. Cross multiplying and solving for x, we get $\frac{100}{7}$.

12) **A) -483**
This is a classic PEMDAS question, so we should do the parentheses first to get $2 \times 18 \div 12 - 216 \div 4(9)$. Then, we do multiplication and division from left to right to get $3 - 486$ (notice that we need to do the $216 \div 4$ first because the parentheses simply signify multiplication, so we still need to go left to right). Evaluating, we get our final answer of -483.

13) **C) 7**
Properties of logarithms says that we can split a logarithm in the form of $log_a b$ into $\frac{log_c a}{log_c b}$. Using this, we can see that the question now telescopes and every term cancels except the first denominator and the last numerator. This turns the problem into $log_2 128 = 7$.

14) **D) 20**
If you and the imp meet each other for the first time in 10 seconds, that means the two of you combined ran around half the column in 10 seconds. Therefore, the two of you combined will run around the full column and meet again in 20 seconds.

15) **D) 1170**
We can see that the points conveniently plot out a right triangle with a hypotenuse from (12, 14) to (2, -10). This means the hypotenuse is 26 units long, so the radius of the circle is 13 units long because the hypotenuse of an inscribed right triangle is always a diameter of the circumscribed circle. That means you are standing 13 units from the generator, which is 390 yards, which is 1170 feet.

16. **D) 104**
The fraction of marbles that are blue in the urn at any given time is $\frac{\text{\# of blue marbles}}{\text{total \# of marbles}}$. From this, we can tell that we only want to add blue marbles. This is because adding red marbles would just decrease the percentage of blue marbles and get us farther away from our goal of 85%.

Let's call the amount of added blue marbles B. We want the percentage of blue marbles in the urn after adding the new marbles to be at least 85%, so we can create the inequality

$$\frac{10 + B}{30 + B} \geq 0.85 = \frac{17}{20}$$

We can cross multiply and move all variables to one side to get

$$20(10 + B) \geq 17(30 + B)$$
$$200 + 20B \geq 510 + 17B$$
$$B \geq \frac{310}{3} = 103.\bar{3}$$

The least whole number of marbles that satisfies this inequality is 104 blue marbles.

17. **C)** $\mathbf{\frac{11}{3}}$
Going step by step, we first add 4, giving us $n + 4$.
Multiplying by 3 gives $3(n + 4) = 3n + 12$.
Subtracting 1 gives $3n + 11$.
Dividing by 3 gives $\frac{3n+11}{3} = n + \frac{11}{3}$.
Subtracting n gives $n + \frac{11}{3} - n = \frac{11}{3}$.

18. **A)** $\mathbf{\frac{3}{2}}$
If we want to take in 360 liters of oxygen and oxygen is 20% (or 1/5) of the volume of air, we know that the volume of air that you will have to breathe in is $5(360)$ liters. To make it easier to divide later, we will leave that unsimplified.

Each breath is 500 milliliters of air, which is equivalent to 0.5 liters of air.
$$\frac{5(360)\text{ liters}}{1} \times \frac{1\text{ breath}}{0.5\text{ liters}} = 3600\text{ breaths}$$

You take 40 breaths per minute and there are 60 minutes in 1 hour, so
$$\frac{3600\text{ breaths}}{1} \times \frac{1\text{ minute}}{40\text{ breaths}} \times \frac{1\text{ hour}}{60\text{ minutes}} = \frac{3}{2}\text{ hours}$$

19. **B) 15**
Let's call the number of beetles B, giraffes G, and spiders S. Each of these animals has one head, so our false equation for the number of total heads is
$$S + B - G = 8$$
Spiders have 8 legs, beetles have 6, and giraffes have 4, so our false equation for the number of legs is
$$8S + 6B - 4G = 78$$

To make things simpler, we will divide both sides by 2 to get
$$4S + 3B - 2G = 39$$

We are left with the following system of equations:
$$S + B - G = 8$$
$$4S + 3B - 2G = 39$$

Although this seems difficult given that we have three variables and only two equations, we know that we are looking for $S + G$ (the total number of red critters), so we will try to cancel out the B coefficient by multiplying the first equation by 3 and subtracting the two resulting equations. This gives:
$$(4S + 3B - 2G) - 3(S + B - G) = 39 - 3(8)$$
$$(4S - 3S) + (3B - 3B) + (-2G + 3G) = 39 - 24$$
$$S + G = 15$$

Spiders and giraffes are our only red critters, so our final answer is 15.

20. **B) 100**
Call the current age of the blue door B and the current age of the red door R. The problem tells us that

$$B = \frac{1}{2}R$$

in the second sentence.

In 100 years, the blue door will be $B + 100$ years old and the red door will be $R + 100$ years old. Therefore, the third sentence tells us that

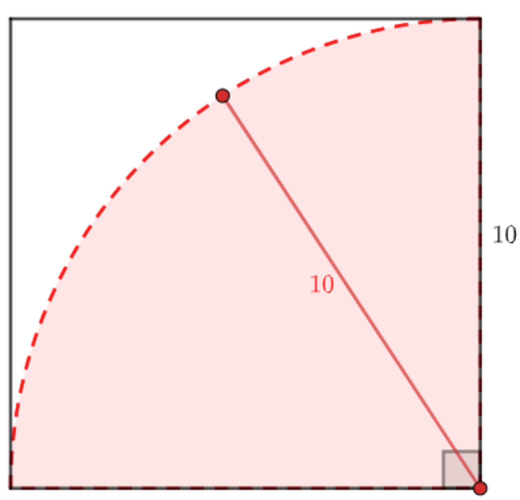
$$B + 100 = \frac{2}{3}(R + 100)$$

Using substitution with $R = 2B$, we get that

$$B + 100 = \frac{2}{3}(2B + 100)$$
$$3(B + 100) = 2(2B + 100)$$
$$3B + 300 = 4B + 200$$
$$B = 100$$

The blue door is 100 years old.

21. **B) 25π**
Refer to the below picture:

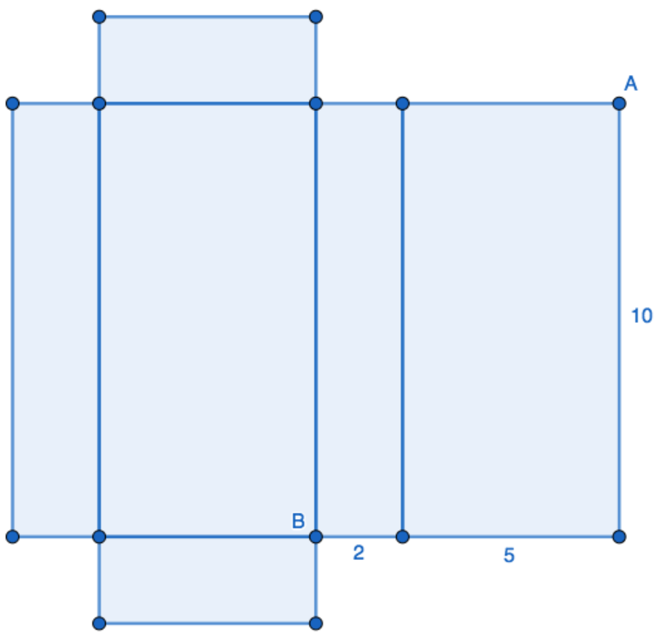


Since the area of the square room is 100 square feet, each side of the room has length 10 feet. The end of the leash is attached to the bottom right corner of the square room, so the area the dog can roam is the shaded red region. This shaded red region is a circular sector with central angle 90 degrees and radius 10. Therefore, the area of the region is simply ¼ of the area of a circle with radius 10:

$$\frac{1}{4}(10^2)\pi = 25\pi$$

22. B) $\sqrt{149}$

Refer to the below picture of the prism unfolded as a net.



Now it is clear that the shortest distance from point A to B is the straight-line distance between the two points. By the Pythagorean theorem, we can see that that is

$$\sqrt{(2+5)^2 + 10\hat{}2} = \sqrt{149}$$

23. **D) 2**

The bridge is 1 mile long, which is 5280 feet. We are looking for the sum of the following sequence through the 5,280th term:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

The 5,280th term of the sequence is

$$1\left(\frac{1}{2}\right)^{5279}$$

This is an extremely, extremely small number. Since it is so small, we know that the sum of the series up through this point is very close to the sum of a convergent geometric series with ratio ½ and first term 1, which is

$$\frac{1}{1-\frac{1}{2}} = 2$$

The exact sum of the series up to the 5,280th term is so close to 2 that the difference is negligible.

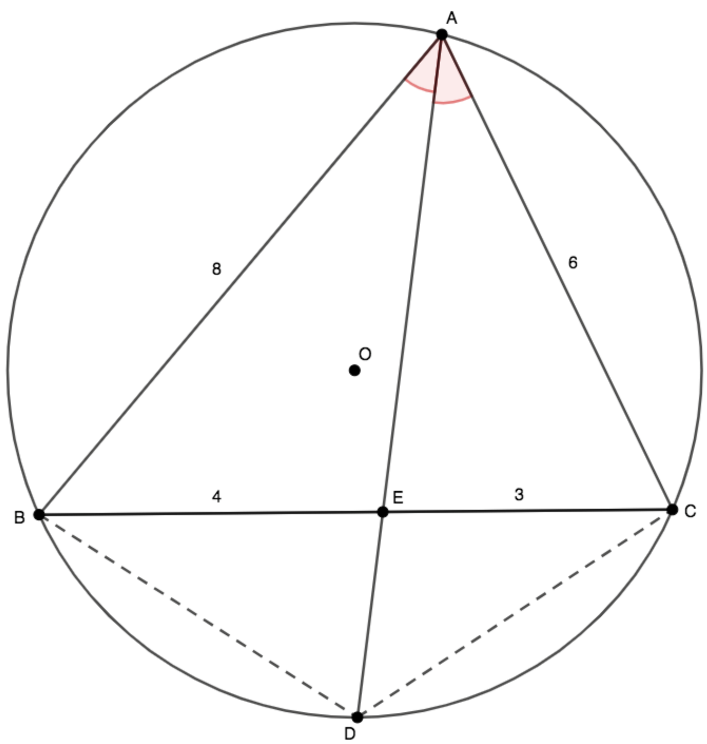
24. **C) 5**

We solve this system of equations by adding them together to eliminate the $y -$ variable.

$$(2x + y) + (3x - y) = 10 + 15$$
$$5x = 25$$
$$x = 5$$

25. **D) 1**

Refer to the below diagram:



Notice that $\frac{BE}{AB} = \frac{CE}{AC} = \frac{1}{2}$. By Angle Bisector Theorem, we can see that $\overline{AE}$ is an angle bisector and $\angle EAC$ and $\angle EAB$ (the two red angles) have the same measure. This also tells us that the measures of arcs BD and CD are equal. Since these two arcs are of equal measure, their corresponding chords $\overline{BD}$ and $\overline{CD}$ are of equal length, and $\frac{CD}{BD} = 1$.

26. **C)** $\frac{2}{3}$

To solve this question, we need to consider all our cases. The following displays all 9 cases:

Your Choice	Winning Door	Revealed Door	Probability to win
A	A	B	?
A	B	C	?
A	C	B	?
B	A	C	?
B	B	A	?
B	C	A	?
C	A	B	?
C	B	A	?
C	C	A	?

Now, we need to figure out an optimal strategy. Clearly, if the situation is like the one in the second row of the table: (A, B, C) (Notation: (Your Choice, Winning Door, Revealed Door)), switching guarantees a win because you now know the correct door is B, otherwise Mr. Lu would've opened B instead. The same goes for (B, A, C) and (C, A, B).

The other 6 cases are identical to the traditional Monty Hall problem. You picked a door, Mr. Lu revealed a door, and that door that is revealed gave you no more information than if the alphabet rule wasn't there. The optimal strategy to the Monty Hall problem is to always switch because the probability of winning is $\frac{2}{3}$ if you switch. Note that the sum of the probabilities of winning for our 6 cases is not $\frac{2}{3}$ overall because there are many more cases in the traditional Monty Hall problem that we neglect in this one.

Here is a brief explanation of why the overall probability of winning by switching in the traditional Monty Hall problem is $\frac{2}{3}$. If you initially pick a door that does not have the prize behind it and switch, you will win. If you choose the door with the prize and switch, you will lose. Therefore, given that you always switch, the probability of winning is the same as the probability that you initially picked a non-prize door: $\frac{2}{3}$. The probability of winning given that you decide NOT to switch is $\frac{1}{3}$; therefore, switching is preferable.

So, we can see the optimal strategy is to always switch, and if we fill out our table, we get:

Your Choice	Winning Door	Revealed Door	Probability to win
A	A	B	0
A	B	C	1
A	C	B	1
B	A	C	1
B	B	A	0
B	C	A	1
C	A	B	1
C	B	A	1
C	C	A	0

From this table, we can see that the probability of winning is $\frac{6}{9} = \frac{2}{3}$.

27. **A)** $\mathbf{\frac{30}{11}}$
The gap between two consecutive numbers on an analog clock is $\frac{360°}{12} = 30°$. At 6:30 PM, the hour hand is halfway between the "6" and "7" marks on an analog clock. The minute hand is exactly at the "6" mark. The minute and hour hand are thus $\frac{30°}{2} = 15°$ apart at 6:30 PM. The hour hand moves at a rate of $\frac{1}{2}°$ per minute, and the minute hand moves at a rate of 6° per minute. Therefore, the gap between the minute and hour hand closes at a rate of $6 - \frac{1}{2} = \frac{11}{2}°$ per minute.

It would thus take $15 \div \frac{11}{2} = \frac{30}{11}$ minutes for the hands to meet.

28. **E) 27**
A "transition" occurs at every root of the function except for any double roots. This function is factorable:

$$\begin{aligned} f(x) &= x^4 - 7x^3 + 16x^2 - 12x \\ &= x(x^3 - 7x^2 + 16x - 12) \\ &= x(x-2)^2(x-3) \end{aligned}$$

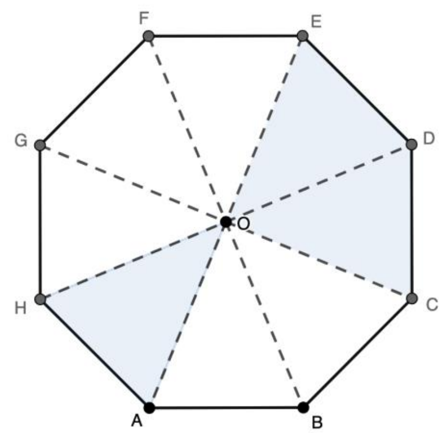
Therefore, the roots are at $x = 0, 2,$ and 3. Since 2 is a double root, 0 and 3 are the only transition points.

$$0^3 + 3^3 = 27$$

29. **C) (4)**
Step (4) is not a logical progression from the previous step.
This is because by order of operations, we know that

$$\sqrt{(-1)^2} = \sqrt{1} = 1 \neq -1$$

30. **B)** $\mathbf{6\sqrt{2}+6}$
Refer to the following diagram:



Regions AHO and OCDE are three of eight equal-sized sectors of the octagon, so their area is $\frac{3}{8}$ of the total area of the octagon. We can now just solve

$$\frac{3}{8}\left(16+16\sqrt{2}\right)=6+6\sqrt{2},$$

which is the area of the intended region.