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SOLUTIONS

1) Convert everything to base 2: \( \log_4(256) \cdot \log_8(32768) \cdot \log_2(32) = \frac{\log_2^{28}}{\log_2^2} \cdot \frac{\log_2^{215}}{\log_2^3} \cdot \frac{\log_2^5}{\log_2^{-1}} = 4 \cdot 5 \cdot -5 = -100. \)  

2) Convert the logs to base 3 to get \([\log_9(x)]^2 - \log_{\frac{1}{27}}(x) + 5 = 0 \rightarrow \left[ \frac{1}{2} \log_3 x \right]^2 - 3 \log_3 x + 5 = 0 \rightarrow \frac{1}{4} (\log_3 x)^2 + 3 \log_3 x + 5 = 0. \) If the solutions are \( a \) and \( b \), then we can find the sum of the solutions in \( \log_3 x \), then use the product property to get \( \log_3 a + \log_3 b = \log_3 ab \). The sum of the solutions is \( \frac{-(3)}{4} = 12 \), so our \( \log_3 ab = 12 \) and \( ab = 3^{12} \).  

3) The maximum value of a logistic function is just the numerator 13.  

4) Examining the first inequality, we have \( 2^{16} < 3^x \rightarrow x > 16 \log_2 3 \rightarrow x > 10.096 \). Examining the right side we have \( 3^x < 2^{32} \rightarrow x < 32 \log_3 2 \rightarrow x < 20.192 \). We need the number of integers between 10.096 and 20.192 which is 10.  

5) Statement I is incorrect, as it includes \( x = 3 \) which would make the argument of the logarithm 0, making it undefined. Statement 2 is correct, as all log functions have range All Reals. Statement III is true: simply switch \( y \) and \( x \) then solve for \( y \).  

6) The sum is a geometric series with first term 1 and common ratio \( 2^x \). The sum is equal to \( \frac{1}{1-2^x} = \frac{2}{2} \). Rearranging, we have \( 2^x = \frac{1}{3} \) which leads to \( x = -\log_2 3 \).  

7) Add up the exponent, (or more easily, just the last two digits of each value in the exponent) and notice it ends in 09, which is 1 mod 4, corresponding to \( i \).  

8) It is a fun fact that \( |z^n| = |z|^n \), so we just need to find the magnitude of \( 3 + 4i \) then raise it to the 4th power. We have \( |3 + 4i| = \sqrt{3^2 + 4^2} = 5 \), so \( 5^4 = 625 \).  

9) If you convert the second to a power of 2, we need \( 2|x^2-1| = 2^2 - 2x^2 \), which means the exponents must be the same. Setting up the requisite 2 cases gives us \( x^2 - 1 = 2 - 2x^2 \) which has solutions \( x = \pm 1 \) and \( x^2 - 1 = 2x^2 - 2 \), which also has solutions \( x = \pm 1 \). Any time you have an absolute value equation where one side can be negative, you must check your solutions, though in this case, both work meaning there are 2 points of intersection.
10) Note that \( \log 5 = \log \frac{10}{2} = \log 10 - \log 2 = 1 - \log 2 = 1 - .301 = 0.699 \). Then \( \log 3125 = \log 5^5 = 5 \log 5 = 5 \cdot 0.699 = 3.495 \) B

11) It is useful to know that \( \log 3 < \log 2000 < 4 \), and it's approximately \( \log 2 \cdot 1000 = \log 2 + \log 1000 = 3.301 \). We simplify \( 2000 \cdot 5 \cdot \log 2000 = 10,000 \cdot 3.301 \approx 33,000 \), which is in the third interval. C

12) We can compare the first two numbers first. \( 6^{99} \) vs \( 7^{75} \), let's manipulate the power of 7 to see \( 7^{75} = \left(\frac{1}{6}\right)^75 \). Cancelling a factor of \( 6^{75} \), we're now comparing \( 6^{24} \) with \( 7^{3} \cdot \left(\frac{1}{6}\right)^3 = \frac{343}{216} \). Since both sides have a n exponent of 24, we can divide both sides by that term and combine to get \( \frac{343}{216} \) compared with \( \frac{343}{216} \). The LHS is clearly larger than the RHS, so the original LHS was larger than the RHS. Now we compare \( 6^{99} \) and \( 8^{50} \). Simplifying \( 8^{50} = 2^{150} \), then cancelling a factor of \( 2^{99} \), we have \( 3^{99} > 2^{51} \) which is clear as day. A

13) Let \( u = e^x \). \( 4u + \frac{5}{u} = 9 \) \( \rightarrow \) Multiply both sides by \( u \rightarrow 4u^2 + 5 = 9u \rightarrow 4u^2 - 9u + 5 = 0 \rightarrow (4u-5)(u-1) = 0 \rightarrow \\
\begin{align*}
4u-5 &= 0 \text{ and } u-1 = 0 \rightarrow u = 5/4 \text{ and } u = 1. \\
\end{align*}
Substitute back in \( e^x = \frac{5}{4} \rightarrow \ln e^x = \ln \frac{5}{4} \rightarrow x = \ln \frac{5}{4} \) and \( e^x = 1 \rightarrow x = 0. \)

\[ \ln \frac{5}{4} + 0 = \ln \frac{5}{4} \] E

14) Just examining the equations gives us \( x = 2, 4 \) as solutions but if you graph the two equations you'll see a third, negative solution. C

15) B is the incorrect restriction as \( a = 0 \) gives us an undefined log. B

16) Listing out a few terms we have \( \log t \cdot 1 + \log t \cdot 2 + \log t \cdot 3 + \cdots = 1 + \log_t 5040 \). The LHS is \( \log_t t! \), but the incongruous base and factorial on the right hand side can be explained with \( t = 8 \) and \( \log_8 8 = 1 \) A

17) This sum ‘telescopes’, when you expand each log with the quotient property. \( \log 1 - \log 3 + \log 2 - \log 4 + \log 3 - \log 5 + \cdots + \log 2020 - \log 2022. \) Everything cancels except the \( \log 2 - \log 2021 - \log 2022 = \\
\log \left( \frac{2}{2021 \cdot 2022} \right) = -\log(2021 \cdot 1011) \) A

18) \( a = 1 \) by Geometric series, and \( b = 2 \) by the formula \( \sum_{n=1}^{\infty} \frac{n}{x^n} = \frac{1/x}{(1-1/x)^2} \) with \( x = 2. \) Therefore \( 3^a 5^b = 3^1 5^2 = 75 \) C
19) Playing around a bit, we see that \((1 + i\sqrt{3})^3\) is an integer, as is \((1 + i\sqrt{3})^6\) and so on. Each multiple of 3 will give us an integer so we need the number of multiples of 3 on the interval. \(2022/3 = 674\), but since 2022 is outside the interval we subtract 1 to get 673. B

20) We either need to know all our squares or experiment to find that \(44^2 = 1936 < 2022 < 2025 = 45^2\), so \(n = 44, \) and \(11n = 484\) with a digital sum of 16 E

21) There are three cases where the LHS could be 1. Case 1: the exponent is equal to 0. \(x^2 + 4x + 4 \rightarrow x = -2\). Case 2: the base is equal to 1 \(x^2 - 9x + 15 = 1 \rightarrow x^2 - 9x + 14 = 0 \rightarrow x = 2, 7\). Case 3: Base equals negative 1 and exponent is even. This case does not have any solutions, as the only \(x - \) values that make the base equal to -1 are irrational and do not make the exponent an even integer. So, \(-2 + 2 + 7 = 7\) B

22) Each part has a cycle of 4 units digits, corresponding to the exponent. \(1^x = 1\) for all real \(x\). \(2^x\) ends in 2, 4, 8, or 6. \(3^x\) ends in 3, 9, 7, or 1. \(4^x\) ends in 4 or 6. We’re looking at the second position in each of these lists, as \(2022/4\) has remainder 2, so we get \(1 + 4 + 9 + 6\) which ends in a 0 A

23) \(9! = 362,880\) C

24) Continuously compounded interest gains the most interest over time. D

25) Using the formula from the previous question, we have \(A = 10000(1 + .1)^4 = 10000 \left(\frac{11}{10}\right)^4 = 10000 \left(\frac{14641}{10000}\right) = 14641\). B

26) This is a geometric series, and we need \(1 + 3 + 3^2 + 3^3 + 3^4 + 3^5 + 3^6 = \frac{1 - 3^7}{1 - 3} = \frac{-2186}{-2} = 1093\). C

27) The LHS of this equation can be written as \((2^x + 3^x)^3 = 13^3\) which means \(2^x + 3^x = 13\), which by inspection has solution \(x = 2\). D

28) Using the Pythagorean Theorem, we have \(x^2 + (\ln x)^2 = 25 \rightarrow \ln x = \sqrt{25 - x^2}\). The question asks how many \(x\) values satisfy this equation, and when viewed geometrically, this is asking how many times does the graph of the natural log intersect a half circle of radius 5? That is exactly once. While solving, you need to take a square root, so you may be inclined to include a \(\pm\), but the value of \(\ln x\) must be positive as it is the length of a leg of a triangle. B
29) Start by setting \( a = \sqrt{x} \) and letting \( \sqrt{11 + 4\sqrt{7}} = \sqrt{x} + \sqrt{b} \). Square both sides to get \( 11 + 4\sqrt{7} = x + b + 2\sqrt{xb} \). Setting \( x + b = 11 \) and \( 4\sqrt{7} = 2\sqrt{28} = xb \), we can see that \( x = 4 \) and \( b = 7 \) (not the other way around because it was assumed earlier that \( x \) could be simplified under a radical). This leads to \( a = 2 \) and \( b = 7 \) so \( b^2 - a^2 = 49 - 4 = 45 \)**B**

30) Write \( f(x) = (x - r_1)(x - r_2)(x - r_3)^2 \) for some \( r_1, r_2, r_3 \in \mathbb{C} \). Then \( f(x)^3 = (x - r_1)^3(x - r_2)^3(x - r_3)^6 \), which corresponds to option **A**.