ANSWERS

1) B	11) C	21) B
2) C	12) A	22) A
3) C	13) E (ln(5/4))	23) C
4) D	14) C	24) D
5) C	15) B	25) B
6) B	16) A	26) C
7) A	17) A	27) D
8) D	18) C	28) B
9) C	19) B	29) B
10) B	20) E (16)	30) A

SOLUTIONS

1) Convert everything to base 2: $\log_4(256)\cdot\log_8(32768)\cdot\log_{\frac{1}{2}}(32) = \frac{\log 2^8}{\log 2^2}\cdot\frac{\log 2^{15}}{\log 2^3}\cdot\frac{\log 2^5}{\log 2^{-1}} = 4*5*-5 = -100$. **B**

2) Convert the logs to base 3 to get $[\log_9(x)]^2 - \log_{\frac{1}{27}}(x) + 5 = 0 \rightarrow \left[\frac{1}{2}\left(\log_3 x\right)\right]^2 - \left(-3\log_3 x\right) + 5 = 0 \rightarrow$ $\frac{1}{4}(\log_3 x)^2 + 3\log_3 x + 5 = 0$. If the solutions are a and b, then we can find the sum of the solutions in $\log_3 x$, then use the product property to get $\log_3 a + \log_3 b = \log_3 ab$. The sum of the solutions is $\frac{(3)}{\frac{1}{4}} = 12$, so our $\log_3 ab = 12$ and $ab = 3^{12}$. **C**

3) The maximum value of a logistic function is just the numerator 13. **C**

4) Examining the first inequality, we have $2^{16} < 3^x \rightarrow x > 16\log_3 2 \rightarrow x > 10.096$. Examining the right side we have $3^x < 2^{32} \rightarrow x < 32\log_3 2 \rightarrow = x < 20.192$. We need the number of integers between 10.096 and 20.192 which is 10. **D**

5) Statement I is incorrect, as it includes $x = 3$ which would make the argument of the logarithm 0, making it undefined. Statement 2 is correct, as all log functions have range All Reals. Statement III is true: simply switch y and x then solve for y. **C**

6) The sum is a geometric series with first term 1 and common ratio 2^x. The sum is equal to $\frac{1}{1-2^x} = \frac{3}{2}$. Rearranging, we have $2^x = \frac{1}{3}$, which leads to $x = -\log_2 3$. **B**

7) Add up the exponent, (or more easily, just the last two digits of each value in the exponent) and notice it ends in 09, which is 1 mod 4, corresponding to i. **A**

8) It is a fun fact that $|z^n| = |z|^n$, so we just need to find the magnitude of $3+4i$ then raise it to the 4th power. We have $|3+4i| = \sqrt{3^2+4^2} = 5$, so $5^4 = 625$. **D**

9) If you convert the second to a power of 2, we need $2^{|x^2-1|} = 2^{2-2x^2}$, which means the exponents must be the same. Setting up the requisite 2 cases gives us $x^2 - 1 = 2 - 2x^2$ which has solutions $x = \pm 1$ and $x^2 - 1 = 2x^2 - 2$, which also has solutions $x = \pm 1$. Any time you have an absolute value equation where one side can be negative, you must check your solutions, though in this case, both work meaning there are 2 points of intersection. **C**

10) Note that $\log 5 = \log \frac{10}{2} = \log 10 - \log 2 = 1 - \log 2 = 1 - .301 = 0.699$. Then $\log 3125 = \log 5^5 = 5 \log 5 = 5 \cdot 0.699 = 3.495$ **B**

11) It is useful to know that $3 < \log 2000 < 4$, and it's approximately $\log 2 * 1000 = \log 2 + \log 1000 = 3.301$. We simplify $2000 * 5 * \log 2000 = 10{,}000 * 3.301 \approx 33{,}000$ which is in the third interval. **C**

12) We can compare the first two numbers first. 6^{99} vs 7^{75}, let's manipulate the power of 7 to see $7^{75} = \left(6 \cdot \frac{7}{6}\right)^{75} = 6^{75}\left(\frac{7}{6}\right)^{75}$. Cancelling a factor of 6^{75} we're now comparing 6^{24} with $\left(\frac{7}{6}\right)^{75} = \left[\left(\frac{7}{6}\right)^3\right]^{24} \cdot \left(\frac{7}{6}\right)^3 = \frac{343}{216}\left(\frac{343}{216}\right)^{24}$. Since both sides have a n exponent of 24, we can divide both sides by that term and combine to get $\left(\frac{1296}{343}\right)^{24}$ compared with $\frac{343}{216}$. The LHS is clearly larger than the RHS, so the original LHS was larger than the RHS. Now we compare 6^{99} and 8^{50}. Simplifying $8^{50} = 2^{150}$, then cancelling a factor of 2^{99}, we have $3^{99} > 2^{51}$ which is clear as day. **A**

13) Let u = e^x. $4u + \frac{5}{u} = 9$ → Multiply both sides by u → $4u^2 + 5 = 9u$ → $4u^2 - 9u + 5 = 0$ → $(4u-5)(u-1) = 0$ →

4u-5 = 0 and u-1 = 0 → u = 5/4 and u = 1. Substitute back in $e^x = \frac{5}{4} \rightarrow \ln e^x = \ln\frac{5}{4} \rightarrow x = \ln\frac{5}{4}$ and $e^x = 1 \rightarrow x = 0$.

$\ln\frac{5}{4} + 0 = \ln\frac{5}{4}$ **E**

14) Just examining the equations gives us $x = 2, 4$ as solutions but if you graph the two equations you'll see a third, negative solution. **C**

15) B is the incorrect restriction as $a = 0$ gives us an undefined log. **B**

16) Listing out a few terms we have $\log_t 1 + \log_t 2 + \log_t 3 + \cdots = 1 + \log_t 5040$. The LHS is $\log_t t!$, but the incongruous base and factorial on the right hand side can be explained with $t = 8$ and $\log_8 8 = 1$ **A**

17) This sum 'telescopes', when you expand each log with the quotient property. $\log 1 - \log 3 + \log 2 - \log 4 + \log 3 - \log 5 + \cdots + \log 2020 - \log 2022$. Everything cancels except the $\log 2 - \log 2021 - \log 2022 = \log\left(\frac{2}{2021*2022}\right) = -\log(2021 * 1011)$ **A**

18) $a = 1$ by Geometric series, and $b = 2$ by the formula $\sum_{n=1}^{\infty} \frac{n}{x^n} = \frac{1/x}{(1-1/x)^2}$ with $x = 2$. Therefore $3^a 5^b = 3^1 5^2 = 75$ **C**

19) Playing around a bit, we see that $\left(1+i\sqrt{3}\right)^3$ is an integer, as is $\left(1+i\sqrt{3}\right)^6$ and so on. Each multiple of 3 will give us an integer so we need the number of multiples of 3 on the interval. $2022/3 = 674$, but since 2022 is outside the interval we subtract 1 to get 673. **B**

20) We either need to know all our squares or experiment to find that $44^2 = 1936 < 2022 < 2025 = 45^2$, so $n = 44$, and $11n = 484$ with a digital sum of 16 **E**

21) There are three cases where the LHS could be 1. Case 1: the exponent is equal to 0. $x^2 + 4x + 4 \rightarrow \boldsymbol{x = -2}$. Case 2: the base is equal to 1. $x^2 - 9x + 15 = 1 \rightarrow x^2 - 9x + 14 = 0 \rightarrow \boldsymbol{x = 2, 7}$. Case 3: Base equals negative 1 and exponent is even. This case does not have any solutions, as the only $x -$ values that make the base equal to -1 are irrational and do not make the exponent an even integer. So, $-2 + 2 + 7 = 7$ **B**

22) Each part has a cycle of 4 units digits, corresponding to the exponent. $1^x = 1$ for all real *x*. 2^xends in 2, 4, 8, or 6. 3^xends in 3, 9, 7, or 1. 4^x ends in 4 or 6. We're looking at the second position in each of these lists, as 2022/4 has remainder 2, so we get $1 + 4 + 9 + 6$ which ends in a 0 **A**

23) $9! = 362{,}880$ **C**

24) Continuously compounded interest gains the most interest over time. **D**

25) Using the formula from the previous question, we have $A = 10000(1 + .1)^4 = 10000\left(\frac{11}{10}\right)^4 = 10000\left(\frac{14641}{10000}\right) = 14641$. **B**

26) This is a geometric series, and we need $1 + 3 + 3^2 + 3^3 + 3^4 + 3^5 + 3^6 = \frac{1-3^7}{1-3} = \frac{-2186}{-2} = 1093$. **C**

27) The LHS of this equation can be written as $(2^x + 3^x)^3 = 13^3$ which means $2^x + 3^x = 13$, which by inspection has solution $x = 2$. **D**

28) Using the Pythagorean Theorem, we have $x^2 + (\ln x)^2 = 25 \rightarrow \ln x = \sqrt{25 - x^2}$. The question asks how many x values satisfy this equation, and when viewed geometrically, this is asking how many times does the graph of the natural log intersect a half circle of radius 5? That is exactly once. While solving, you need to take a square root, so you may be inclined to include a $\pm$, but the value of $\ln x$ must be positive as it is the length of a leg of a triangle. **B**

29) Start by setting $a = \sqrt{x}$ and letting $\sqrt{11 + 4\sqrt{7}} = \sqrt{x} + \sqrt{b}$. Square both sides to get $11 + 4\sqrt{7} = x + b + 2\sqrt{xb}$. Setting $x + b = 11$ and $4\sqrt{7} = 2\sqrt{28} = xb$, we can see that $x = 4$ and $b = 7$ (not the other way around because it was assumed earlier that x could be simplified under a radical). This leads to $a = 2$ and $b = 7$ so $b^2 - a^2 = 49 - 4 = 45$ **B**

30) Write $f(x) = (x - r_1)(x - r_2)(x - r_3)^2$ for some $r_1, r_2, r_3 \in \mathbb{C}$. Then $f(x)^3 = (x - r_1)^3(x - r_2)^3(x - r_3)^6$, which corresponds to option **A**.