

Important Instructions for this Test: The answer “E: NOTA” represents that “None of These Answers” is correct. The notation $\mathbb{R}^n$ or $\mathbb{C}^n$ represents the set of length- n vectors with real or complex components respectively. The notation $\mathbb{R}^{m \times n}$ and $\mathbb{C}^{m \times n}$ represents the set of $m \times n$ matrices. Unless otherwise specified, all vectors are column matrices, although we may use more in-line friendly $\langle a, b, c \rangle$ notation.

1. Legosi has overheard a few different statements about matrices but doesn't know how many are true!

I: If A is a 2×3 matrix, then $A^T A$ is a 2×2 matrix.

II: If A^2 is well-defined, then A is a square matrix.

III: If $A^2 - A$ is the zero matrix, then either A or $A - I$ is the zero matrix.

IV: If A and B are matrices such that AB is well-defined, then BA is also well-defined.

How many of these statements are true? (“OTA” answer choices represent that “4 Of The Above” are true, etc., all the way down to “None Of The Above” are true.)

A: 4OTA B: 3OTA C: 2OTA D: 1OTA E: NOTA

2. Compute $2\langle 2, -3 \rangle - 3\langle -1, 1 \rangle$.

A: $\langle -1, -9 \rangle$ B: $\langle 7, -9 \rangle$ C: $\langle -1, -3 \rangle$ D: $\langle 7, -3 \rangle$ E: NOTA

3. Find the vector $\mathbf{v}$ such that $2(\mathbf{v} + \langle -2, 1 \rangle) = 3(\langle 2, -1 \rangle - \mathbf{v})$.

A: $\langle 2, -1 \rangle$ B: $\langle -1, 2 \rangle$ C: $\langle 1, -2 \rangle$ D: $\langle -2, 1 \rangle$ E: NOTA

4. If $\mathbf{v}$ and $\mathbf{w}$ are vectors such that $\mathbf{v}(\mathbf{w}^T) = \begin{pmatrix} 3 & -6 & -3 \\ 1 & -2 & -1 \\ -2 & 4 & 2 \end{pmatrix}$, compute $\mathbf{v} \cdot \mathbf{w}$.

A: -7 B: -4 C: 0 D: 3 E: NOTA

Please Use the Following Information to Answer Questions 5 to 6:

A function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called a *linear transformation* if for any vectors $\mathbf{v}_1, \mathbf{v}_2$ in $\mathbb{R}^n$ and any scalars c_1, c_2 in $\mathbb{R}$, $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$.

5. If $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfies $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, compute $T\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

A: $\langle 2, -3 \rangle$ B: $\langle -4, -1 \rangle$ C: $\langle -1, 3 \rangle$ D: $\langle 8, 1 \rangle$ E: NOTA

6. If $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfies $T\begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} -3 \\ 9 \end{pmatrix}$ and $T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$. Given that T^{-1} exists, find $T^{-1}\begin{pmatrix} 5 \\ 12 \end{pmatrix}$.

A: $\langle 13, 17 \rangle$ B: $\langle -9, 21 \rangle$ C: $\langle 5, 12 \rangle$ D: $\langle 9, 11 \rangle$ E: NOTA

7. The definition of linear transformation generalizes to functions to/from matrices! How many of the following functions are linear transformations?

I: $f_1 : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^{m \times n}$ defined such that $f_1(A) = A^T$.

II: $f_2 : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}$ defined by $f_2(A) = \text{tr}(A)$. (Here ‘tr’ represents the *trace* of a matrix)

III: $f_3 : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ defined by $f_3(A) = \det(A)$.

IV: $f_4 : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n \times n}$ defined by $f_4(A) = A^2$.

A: 4OTA B: 3OTA C: 2OTA D: 1OTA E: NOTA

Please Use the Following Information to Answer Questions 8-9:

A *basis* of $\mathbb{R}^n$ is a list of n vectors such that every vector in $\mathbb{R}^n$ has a unique representation as a linear combination of vectors in the basis. For example, the list (we're using curly braces but these are ordered lists on this test) $\{\langle 1,0,0 \rangle, \langle 0,1,0 \rangle, \langle 0,0,1 \rangle\}$ is a basis of $\mathbb{R}^3$ (in fact, it is called the *standard basis* of $\mathbb{R}^3$, and other standard bases are analogous; take a moment to convince yourself it is indeed a basis).

Given a vector $\mathbf{v}$ and a basis $L = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, the *coordinate representation of $\mathbf{v}$* with respect to L , denoted $[\mathbf{v}]_L$, is the vector $\langle c_1, \dots, c_n \rangle$, where $c_1, \dots, c_n$ are the unique values such that $\mathbf{v} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$.

Indeed, the standard way we write vectors is actually the coordinate representation of the vector with respect to the standard basis!

8. For which of the following vectors $\mathbf{v}$ is $\{\langle 1, -2 \rangle, \mathbf{v}\}$ *not* a basis of $\mathbb{R}^2$?

- A: $\langle -2, 1 \rangle$ B: $\langle -2, 4 \rangle$ C: $\langle 1, 2 \rangle$ D: $\langle 0, -2 \rangle$ E: NOTA

9. Given that $L = \{\langle 0,1,1 \rangle, \langle 1,0,1 \rangle, \langle 1,1,0 \rangle\}$ is a basis of $\mathbb{R}^3$, if $[\langle 3, -4, 5 \rangle]_L = \langle a, b, c \rangle$, find a .

- A: 2 B: -3 C: -1 D: 6 E: NOTA

Please Use the Following Information to Answer Questions 10-11:

Given a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and two bases, $L_1 = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of $\mathbb{R}^n$ and L_2 of $\mathbb{R}^m$, the *matrix representation of T* with respect to L_1 and L_2 , denoted $[T]_{L_1}^{L_2}$, is the matrix M defined column-wise such that the n th column of M is $[T(\mathbf{v}_n)]_{L_2}$.

If either of L_1 or L_2 is not specified in the notation, the corresponding basis is the standard basis.

10. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation such that $T(\mathbf{v})$ is the reflection of $\mathbf{v}$ over the line $y = 2x$ (convince yourself this is a linear transformation; the fact that $y = 2x$ passes through the origin is important). Compute $[T]$ (so in this case L_1 and L_2 are both the standard basis of $\mathbb{R}^2$, $\{\langle 1,0 \rangle, \langle 0,1 \rangle\}$).

- A: $\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}$ B: $\begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$ C: $\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$ D: $\begin{pmatrix} \frac{2}{5} & \frac{4}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}$ E: NOTA

11. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation such that

$$[T] = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Given that $L = \{\langle 1,1,1 \rangle, \langle 1,1,0 \rangle, \langle 1,0,0 \rangle\}$ is a basis of $\mathbb{R}^3$, compute $[T]_L^L$.

- A: $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ C: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ E: NOTA
 B: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ D: $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

12. Find the sum of the components of $\langle 1, 1 + i, 1 - i \rangle \times \langle i, -i, -1 \rangle$.

- A: $-1 - 3i$ B: $3 - i$ C: $-1 - i$ D: 0 E: NOTA

Please Use the Following Information to Answer Questions 13-14:

Given a matrix A , an *eigenvalue* of A is a value λ such that $\det(A - \lambda I) = 0$. Each eigenvalue has at least one corresponding *eigenvector*, which is a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.

For the following two problems, let $A = \begin{pmatrix} -2 & 2 & -2 \\ -6 & 5 & -6 \\ -2 & 1 & -2 \end{pmatrix}$.

13. Find the sum of the squares of the eigenvalues of A .

- A: 2 B: 10 C: 5 D: 6 E: NOTA

14. If λ is the largest eigenvalue of A , find the absolute value of the sum of the coordinates of an eigenvector of A with length 1 and corresponding eigenvalue λ .

- A: $\frac{3\sqrt{5}}{5}$ B: $\frac{\sqrt{5}}{5}$ C: $\sqrt{2}$ D: 0 E: NOTA

Please Use the Following Information to Answer Questions 15 to 18:

Vector projections can be used to compute distances between points, lines, and planes!

15. Find the distance between the point $(-3, 6)$ and the line $2x + y = 1$.

- A: 1 B: $\frac{\sqrt{5}}{5}$ C: $\frac{3\sqrt{5}}{5}$ D: $\frac{2\sqrt{5}}{5}$ E: NOTA

16. Find the distance between the point $(1, -1, 1)$ and the plane $2x + y - 2z = 1$.

- A: 1 B: 2 C: $\frac{2}{3}$ D: $\frac{1}{3}$ E: NOTA

17. Find the distance between the point $(1, 2, -1)$ and the line $\frac{x}{2} = \frac{1-y}{2} = -z$.

- A: $\frac{\sqrt{78}}{9}$ B: $\frac{\sqrt{15}}{3}$ C: $\frac{\sqrt{26}}{3}$ D: $\frac{1}{3}$ E: NOTA

18. Find the distance between the lines parameterized by $\ell_1(t) = \langle 1 + t, 2 - t, -t \rangle$ and $\ell_2(t) = \langle 2 - t, -1, t + 1 \rangle$.

- A: $\sqrt{2}$ B: $\frac{\sqrt{2}}{2}$ C: $\frac{2\sqrt{3}}{3}$ D: 2 E: NOTA

19. Find the sum of the entries of the inverse of $\begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix}$.

- A: 1 B: $\frac{47}{16}$ C: $-\frac{49}{16}$ D: -5 E: NOTA

20. Compute the determinant of $\begin{pmatrix} 0 & 2 & 2 & 2 \\ -1 & 0 & 2 & 2 \\ -1 & -1 & 0 & 2 \\ -1 & -1 & -1 & 0 \end{pmatrix}$.

- A: -2 B: -6 C: 6 D: 2 E: NOTA

Please Use the Following Information to Answer Questions 21-24:

A *diagonal* matrix is an $n \times n$ matrix A with zeroes off the main diagonal. Some (but not all) matrices A (which is the standard matrix representation of a transformation $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$) can be *diagonalized*, which means one can find a basis L such that $[T_A]_L^L$ is diagonal.

For the following three problems, let

$$A = \begin{pmatrix} -5 & -3 & 0 \\ 6 & 4 & 0 \\ 14 & 10 & -1 \end{pmatrix},$$

which has eigenvalues $(\lambda_1, \lambda_2, \lambda_3) = (-2, -1, 1)$ with corresponding eigenvectors $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = (\langle -1, 1, 4 \rangle, \langle 0, 0, 1 \rangle, \langle -1, 2, 3 \rangle)$ respectively. To diagonalize A , imagine instead applying T_A to a vector $\mathbf{v}$ in three steps:

1. Write $\mathbf{v}$ as a linear combination of the given eigenvectors of A (you may assume this is possible).
2. Apply T_A to this linear combination, which has the effect of scaling each eigenvector by its corresponding eigenvalue (and linear transformations play nicely with linear combinations).
3. Convert back to standard coordinates (i.e. simplify the linear combination resulting from step 2).

This process suggests we can write $A = S_3 D_2 S_1$, where numbers correspond to the given steps.

21. Which of the following matrices is the value of S_3 ? The desired behavior is to take a vector of the form $\langle a, b, c \rangle$ to $a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3$.

A: $\begin{pmatrix} -2 & -1 & 0 \\ 5 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

C: $\begin{pmatrix} -2 & 5 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

E: NOTA

B: $\begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 4 & 1 & 3 \end{pmatrix}$

D: $\begin{pmatrix} -1 & 1 & 4 \\ 0 & 0 & 1 \\ -1 & 2 & 3 \end{pmatrix}$

22. Which of the following matrices is the value of D_2 ? The desired behavior is to take a vector of the form $\langle a, b, c \rangle$ to $\langle \lambda_1 a, \lambda_2 b, \lambda_3 c \rangle$.

A: $\begin{pmatrix} -2 & -2 & -2 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

C: $\begin{pmatrix} -2 & -1 & 1 \\ -2 & -1 & 1 \\ -2 & -1 & 1 \end{pmatrix}$

E: NOTA

B: $\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -2 & 0 & 0 \end{pmatrix}$

D: $\begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

23. Which of the following matrices is the value of S_1 ? The desired behavior is to take a vector $\mathbf{v}$ to the vector $\langle a, b, c \rangle$, where a, b , and c satisfy $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3$.

A: $\begin{pmatrix} -2 & -1 & 0 \\ 5 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

C: $\begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 4 & 1 & 3 \end{pmatrix}$

E: NOTA

B: $\begin{pmatrix} -2 & 5 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

D: $\begin{pmatrix} -1 & 1 & 4 \\ 0 & 0 & 1 \\ -1 & 2 & 3 \end{pmatrix}$

It takes a bit more work to show this actually fits the definition of diagonalization written above (namely thinking about how to represent $[T_A]_L^L$ as a matrix multiplication and checking these matrices align with that); but if you got these correct, you've just diagonalized A !

24. Use the previous three problems (or don't, I'm not going to tell you how to live your life) to compute the sum of the elements of A^{10} .

A: 12293

B: 6149

C: -12273

D: -6131

E: NOTA

25. Not every matrix is diagonalizable! Which of the following matrices cannot be diagonalized?

A: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ B: $\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$ C: $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ D: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ E: NOTA

26. Even though some matrices are not diagonalizable, we can get pretty close! If instead of requiring the matrix to be diagonal, we also allow 1s directly above the diagonal, we get the *Jordan canonical form (JCF)* of the matrix instead, which has many similar nice properties!

Indeed, for the matrix A defined below, compute the sum of the entries of A^{10} .

$$A = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ -2 & 1 & 1 \end{pmatrix}.$$

A: 3 B: 4 C: -4 D: -3 E: NOTA

27. Note that $\mathbf{v} = \langle 0, -1, 1 \rangle$ is *not* an eigenvector of A in problem 26, i.e. $A\mathbf{v} \neq \lambda\mathbf{v}$ for any λ . However, $\mathbf{v}$ still has some nice properties! Which of the following equations does $\mathbf{v}$ satisfy?

A: $A^2\mathbf{v} = A\mathbf{v}$

C: $A^2\mathbf{v} = \mathbf{v}$

E: NOTA

B: $(I - A)A\mathbf{v} = \mathbf{v}$

D: $(2I - A)A\mathbf{v} = \mathbf{v}$

28. Let $T_A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation that reflects a vector $\mathbf{v}$ over the plane $x + y + z = 0$, and

let $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Evaluate $([T_A]B)^{25}$.

A: $\begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \end{pmatrix}$

C: $\begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \end{pmatrix}$

E: NOTA

B: $\begin{pmatrix} -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$

D: $\begin{pmatrix} -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$

29. Let C_0 be the curve $|x + y| + |x - y| = 2$ (a square of side length 2) as a subset of the xy -plane. Now let C_1 be the curve obtained by projecting each point of C_0 orthogonally onto $x + y + z = 0$. Then let C_2 be the curve obtained by projecting each point of C_1 orthogonally onto $x + y + 2z = 0$. Find the area of C_2 .

A: $\frac{8\sqrt{6}}{9}$

B: $\frac{4\sqrt{6}}{3}$

C: $\sqrt{6}$

D: $\frac{5\sqrt{6}}{4}$

E: NOTA

30. One last one for the road: how many of the following are equal to 2 for the matrix

$$A = \begin{pmatrix} -3 & 0 & 5 \\ 2 & 1 & -1 \\ -1 & 1 & 4 \end{pmatrix}?$$

I: $\text{Tr}(A)$

II: $\det(A)$

III: $\text{rank}(A)$

IV: $\text{rank}(A^T)$

A: 40TA

B: 30TA

C: 20TA

D: 10TA

E: NOTA