Answer Key:

1. D
2. B
3. A
4. D
5. D

6. A
7. C
8. B
9. C
10. B

11. D
12. B
13. C
14. A
15. B

16. C
17. C
18. A
19. C
20. C

21. B
22. D
23. A
24. C
25. A

26. A
27. D
28. A
29. A
30. B

Solutions:

1. D: One is true.

- I: **False**—A^T is 3×2, and a 3×2 matrix times a 2×3 matrix is 3×3, not 2×2.
- II: **True**—Let A be $m \times n$. AA being well-defined means $m = n$, so A is square.
- III: **False**—$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is a counterexample.
- IV: **False**—Suppose A is 1×2, and B is 2×3, then AB is well-defined but BA is not.

2. B: $\langle 2(2) - 3(-1), 2(-3) - 3(1) \rangle = \boxed{7, -9}$.

3. A: Expand to get $2\boldsymbol{v} + \langle -4,2 \rangle = \langle 6, -3 \rangle - 3\boldsymbol{v} \to 5\boldsymbol{v} = \langle 10, -5 \rangle \to \boldsymbol{v} = \boxed{\langle 2, -1 \rangle}$.

4. D: Write $\boldsymbol{v}(\boldsymbol{w}^T) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} (w_1 \quad w_2 \quad w_3) = \begin{pmatrix} v_1 w_1 & v_1 w_2 & v_1 w_3 \\ v_2 w_1 & v_2 w_2 & v_2 w_3 \\ v_3 w_1 & v_3 w_2 & v_3 w_3 \end{pmatrix}$. Then $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 = 3 + (-2) + 2 = \boxed{3}$.

5. D: By the given definition, $T\begin{pmatrix} 2 \\ -3 \end{pmatrix} = T\left(2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-3)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = 2T\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 3T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 3\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \boxed{\langle 8,1 \rangle}$.

6. A: Note that an invertible linear function is itself linear:

$$T^{-1}(c_1 T(\boldsymbol{v}_1) + c_2 T(\boldsymbol{v}_2)) = T^{-1}\big(T(c_1 \boldsymbol{v}_1 + c_2 \boldsymbol{v}_2)\big) = c_1 \boldsymbol{v}_1 + c_2 \boldsymbol{v}_2 = c_1 T^{-1}\big(T(\boldsymbol{v}_1)\big) + c_2 T^{-1}\big(T(\boldsymbol{v}_2)\big).$$

We can note that $\begin{pmatrix} 5 \\ 12 \end{pmatrix} = \frac{7}{3}\begin{pmatrix} -3 \\ 9 \end{pmatrix} + 3\begin{pmatrix} 4 \\ -3 \end{pmatrix}$, so $T^{-1}\begin{pmatrix} 5 \\ 12 \end{pmatrix} = \frac{7}{3}T^{-1}\begin{pmatrix} -3 \\ 9 \end{pmatrix} + 3T^{-1}\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \frac{7}{3}\begin{pmatrix} 3 \\ 6 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \boxed{\langle 13,17 \rangle}$.

7. C: Two are linear transformations.

- I: **Yes**; transposing preserves linear combinations, as corresponding coordinates line up.
- II: **Yes**; the diagonal coordinates line up, so linear combinations behave nicely.
- III: **No**; $4 = \det\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \neq \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2$.
- IV: **No**; $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)^2 \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^2 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

8. B: Let $\boldsymbol{v} = \langle v_1, v_2 \rangle$. This is equivalent to the system of equations $x - 2y = a, v_1 x + v_2 y = b$ having a solution for any (a, b). Note that if $v_1 x + v_2 y$ is not a linear multiple of $x - 2y$ then elimination will always work to give a (unique) solution, so there can only not be a solution when it is a multiple. This means we are looking for a multiple of $\langle 1, -2 \rangle$, which is $\boxed{\langle -2,4 \rangle}$.

9. C: We want $a\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$, which yields the system $b + c = 3, a + c = -4, a + b = 5$. Adding all three equations together and dividing by 2 gives $a + b + c = 2$, so $a = (a + b + c) - (b + c) = 2 - 3 = \boxed{-1}$.

10. B: We need to figure out where this takes the vectors $\langle 1,0\rangle$ and $\langle 0,1\rangle$, for which it suffices to reflect the points $(1,0)$ and $(0,1)$ over the line $y = 2x$. To do this, we find the perpendicular line to $y = 2x$ passing through each of these points, namely $x + 2y = 1$ and $x + 2y = 2$ respectively. These intersect the line $y = 2x$ at $\left(\frac{1}{5}, \frac{2}{5}\right)$ and $\left(\frac{2}{5}, \frac{4}{5}\right)$ respectively. Reflecting $(1,0)$ and $(0,1)$ over these points yields $\left(-\frac{3}{5}, \frac{4}{5}\right)$ and $\left(\frac{4}{5}, \frac{3}{5}\right)$ respectively. Using the given definition gives the matrix $\begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$

11. D: All vectors are written in standard coordinates unless otherwise specified. To compute $[T]_L^L$, we first compute $T\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}3\\2\\1\end{pmatrix}, T\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix}, T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$. We now need to write each of these as a linear combination of the basis vectors. Solving three simple systems of equations gives L coordinate representations of $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ respectively, so $[T]_L^L = \boxed{\begin{pmatrix}1 & 0 & 0\\1 & 1 & 0\\1 & 1 & 1\end{pmatrix}}$.

12. B: Setting up the determinant gives $\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1+i & 1-i \\ i & -i & -1 \end{vmatrix}$. Cofactor expansion along the top row (being careful with signs) gives $\boldsymbol{i}\big((1+i)(-1) - (-i)(1-i)\big) - \boldsymbol{j}\big(1(-1) - (i)(1-i)\big) + \boldsymbol{k}\big(1(-i) - (i)(1+i)\big) = \langle -1 - i + i + 1, -(-1 - i - 1), -i - i + 1\rangle = \langle 0, 2+i, 1-2i\rangle$, giving a sum of $\boxed{3-i}$.

13. C: $\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 2 & -2 \\ -6 & 5-\lambda & -6 \\ -2 & 1 & -2-\lambda \end{vmatrix}$

$$= (-2-\lambda)\big((5-\lambda)(-2-\lambda) - 1(-6)\big) - 2\big(-6(-2-\lambda) - (-2)(-6)\big) - 2\big((-6)(1) - (-2)(5-\lambda)\big)$$
$$= (-2-\lambda)(\lambda^2 - 3\lambda - 4) - 2(6\lambda) - 2(-2\lambda + 4)$$
$$= -\lambda^3 + (-2+3)\lambda^2 + (4 + 6 - 12 + 4)\lambda + (8 - 8)$$
$$= -(\lambda^3 - \lambda^2 - 2\lambda)$$
$$= -\lambda(\lambda + 1)(\lambda - 2).$$

Thus the eigenvalues of A are 0, -1, and 2, the sum of the squares of which is $\boxed{5}$.

14. A: We want to find $\boldsymbol{v}$ such that $A\boldsymbol{v} = 2\boldsymbol{v}$, or $(A - 2I)\boldsymbol{v} = \langle 0,0,0\rangle$. We want

$$\begin{pmatrix} -4 & 2 & -2 \\ -6 & 3 & -6 \\ -2 & 1 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Note that all scalar multiples of the vector we find will work, so without loss of generality (for now, we'll rescale to get length 1 later) let $v_1 = 1$. This results in the equations $-4 + 2v_2 - 2v_3 = 0, -6 + 3v_2 - 6v_3 = 0$, and $-2 + v_2 - 4v_3 = 0$. Solving this system (one of the equations is redundant), gives $v_2 = 2, v_3 = 0$, so the vector we want is a scalar multiple of $\langle 1,2,0\rangle$. Normalizing to length 1 gives $\pm\langle \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}, 0\rangle$. In either case the absolute value of the sum of the coordinates is $\boxed{\frac{3\sqrt{5}}{5}}$.

15. B: We want the perpendicular distance from $(-3,6)$ to the line $2x + y = 1$. We can solve this by just drawing a perpendicular line, but let's do vectors. Pick a point P on the line arbitrarily, say $(0,1)$. The desired distance is the length of the projection of [the vector from $(-3,6)$ to $(0,1)$] onto a vector perpendicular to $2x + y = 1$, say $\langle 2,1\rangle$ (such a vector is perpendicular to $2x + y = 0$ by the dot product characterization of perpendicularity; then parallel lines). The length of the projection is then $\frac{|\langle 2,1\rangle \cdot \langle 3,-5\rangle|}{||\langle 2,1\rangle||} = \frac{1}{\sqrt{5}} = \boxed{\frac{\sqrt{5}}{5}}$.

16. C: We take a similar approach to before. Pick a point on the plane, say $(0,1,0)$, draw the vector from $(1,-1,1)$ to it, and then project onto a vector perpendicular to the plane, say $\langle 2,1,-2\rangle$ (works by same dot product argument as before). Doing so gives $\frac{|\langle -1,2,-1\rangle\cdot\langle 2,1,-2\rangle|}{||\langle 2,1,-2\rangle||} = \boxed{\frac{2}{3}}$.

17. C: We want the perpendicular distance from $(1,2,-1)$ to the line $\langle 0,1,0\rangle + \langle 2,-2,-1\rangle t$. We'd like to do something similar to the previous problems, but have to be careful since there are now many perpendicular directions to the line, and we have to get the right one (before with the plane there was only one perpendicular direction.

Pick a point on the line, say $\langle 0,1,0\rangle$, and draw the right triangle with vertices at this point $(1,2,-1)$ and the projection of $\langle 1,2,-1\rangle$ onto the line (whatever it is…). Notice that the length of this perpendicular is $\sin(\theta)$ times the length of the hypotenuse of this triangle, where θ is the angle between the line and the vector from $\langle 1,2,-1\rangle$ to $\langle 0,1,0\rangle$. We can compute $\cos(\theta) = \frac{\langle -1,-1,1\rangle\cdot\langle 2,-2,-1\rangle}{||\langle -1,-1,1\rangle||\cdot||\langle 2,-2,1\rangle||} = \frac{-1}{3\sqrt{3}}$, so $\sin(\theta) = \frac{\sqrt{26}}{3\sqrt{3}}$. The length of the hypotenuse is $||\langle -1,-1,1\rangle|| = \sqrt{3}$, so our desired length is $\boxed{\frac{\sqrt{26}}{3}}$.

18. A: We want the perpendicular distance between the lines $\langle 1,2,0\rangle + \langle 1,-1,-1\rangle t$ and $\langle 2,-1,1\rangle + \langle -1,0,1\rangle t$. We can use a similar approach to problems 15-16 this time, since there is once again only one perpendicular direction to both lines; we can take a cross product of their direction vectors!

Pick points on each line, namely $(1,2,0)$ and $(2,-1,1)$; the vector connecting them is $\langle 1,-3,1\rangle$. Taking the cross product of the direction vectors gives $\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -1 & -1 \\ -1 & 0 & 1 \end{vmatrix} = \langle -1,0,-1\rangle$; projecting gives $\frac{|\langle 1,-3,1\rangle\cdot\langle -1,0,-1\rangle|}{||\langle -1,0,1\rangle||} = \boxed{\sqrt{2}}$.

19. C: We augment the matrix and do Gaussian elimination. Divide each row by -2 to get

$$\left(\begin{array}{cccc|cccc} 1 & -1/2 & 0 & 0 & -1/2 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1/2 \end{array}\right)$$

Then add $\frac{1}{2}$ of the second, third, and fourth rows to the first, second, and third rows respectively to get

$$\left(\begin{array}{cccc|cccc} 1 & 0 & -1/4 & 0 & -1/2 & -1/4 & 0 & 0 \\ 0 & 1 & 0 & -1/4 & 0 & -1/2 & -1/4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1/2 & -1/4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1/2 \end{array}\right)$$

Then add $\frac{1}{4}$ of the third and fourth rows to the first and second rows respectively to get

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1/2 & -1/4 & -1/8 & -1/16 \\ 0 & 1 & 0 & 0 & 0 & -1/2 & -1/4 & -1/8 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1/2 & -1/4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1/2 \end{array}\right)$$

The RHS is the desired inverse, with a sum of elements equal to $\boxed{-\frac{49}{16}}$.

20. C: We'll do row operations to get the matrix to be upper triangular, and then the determinant is simply the product of the diagonal elements (if you don't know why, consider repeatedly cofactor expanding along the first column). First we'll swap rows 1<->2, 2<->3, 3<->4 (in that order) to get

$$\begin{pmatrix} -1 & 0 & 2 & 2 \\ -1 & -1 & 0 & 2 \\ -1 & -1 & -1 & 0 \\ 0 & 2 & 2 & 2 \end{pmatrix}$$

Then subtract the first row from the second and third to get

$$\begin{pmatrix} -1 & 0 & 2 & 2 \\ 0 & -1 & -2 & 0 \\ 0 & -1 & -3 & -2 \\ 0 & 2 & 2 & 2 \end{pmatrix}$$

Then subtract the second row from the third row and add it twice to the fourth row to get

$$\begin{pmatrix} -1 & 0 & 2 & 2 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -2 & 2 \end{pmatrix}$$

Then subtract twice the third row from the fourth row to get

$$\begin{pmatrix} -1 & 0 & 2 & 2 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

The determinant of this resulting matrix is $(-1)(-1)(-1)(6) = -6$. The row operations we took to get here were three row swaps, multiplying by (-1) and a bunch of additions/subtractions, which don't affect the determinant. The resulting determinant is thus $\boxed{6}$.

21. B: Let the matrix be $\begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}$. Multiplying by $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ gives

$$\langle as_{11} + bs_{12} + cs_{13}, as_{11} + bs_{12} + cs_{13}, as_{11} + bs_{12} + cs_{13} \rangle$$

We need this to equal $\langle av_{11} + bv_{12} + cv_{13}, av_{11} + bv_{12} + cv_{13}, av_{11} + bv_{12} + cv_{13} \rangle$ for all a, b, c, so matching coefficients means $s_{ij} = v_{ij}$ for all relevant i, j. This gives

$$\boxed{\begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 4 & 1 & 3 \end{pmatrix}}.$$

22. D: Let the matrix be $\begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}$. Multiplying by $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ gives

$$\langle as_{11} + bs_{12} + cs_{13}, as_{11} + bs_{12} + cs_{13}, as_{11} + bs_{12} + cs_{13} \rangle$$

We need this to equal $\langle \lambda_1 a, \lambda_2 b, \lambda_3 c \rangle$ for all a, b, c, so $s_{11} = \lambda_1$, $s_{22} = \lambda_2$, and $s_{33} = \lambda_3$ and all other $s_{ij} = 0$. This gives

$$\boxed{\begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}.$$

23. A: We want S_1 to satisfy $S_1\boldsymbol{v} = \langle a, b, c\rangle$ where a, b, c satisfy

$$\begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Multipling both sides by the inverse of the matrix on the LHS gives us exactly what we want, so we want

$$\begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 4 & 1 & 3 \end{pmatrix}^{-1}$$

Cofactor expansion on the second column gives a determinant of $-1\big(-1(2) - 1(-1)\big) = 1$. We then do the usual matrix of minors -> sign pattern -> transpose -> divide by determinant (=1 in this case) to get

$$\begin{pmatrix} -2 & -5 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} -2 & 5 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} -2 & -1 & 0 \\ 5 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \to \boxed{\begin{pmatrix} -2 & -1 & 0 \\ 5 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}}$$

24. C: From the previous three problems, $A = S_3 D_2 S_1$, where $S_1 = S_3^{-1}$. Raising to the 10th power, all of the internal S_1's and S_3's cancel, yielding $A^{10} = S_3 D_2^{10} S_1$. Fortunately diagonal matrices are easy to take powers of, so we get

$$\begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 4 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1024 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & -1 & 0 \\ 5 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 4 & 1 & 3 \end{pmatrix}\begin{pmatrix} -2048 & -1024 & 0 \\ 5 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 2047 & 1023 & 0 \\ -2046 & -1022 & 0 \\ -8184 & -4092 & 1 \end{pmatrix}$$

This gives a sum of $\boxed{-12273}$.

25. A: The process for diagonalization in problems 21-23 works as long as you can find eigenvalues and eigenvectors of the matrix (always true) and that the eigenvectors form a basis (needed for step 1). We now check each of the given matrices

A: $\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$, which only has one eigenvalue, 1. To find the corresponding eigenvector(s), we set $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to y = 0$. Thus there is only one "line" of eigenvectors, which do not form a basis, so $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

B: $\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$ is already diagonal, so we can let $S_1 = S_3 = I$ and be done!

C: $\begin{vmatrix} 2-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)$, which has eigenvalues $\lambda = 1,2$. Each eigenvalue corresponds to at least one eigenvector, and they must be linearly independent (you can't scale the same nonzero vector by 1 and 2 simultaneously), so they form a basis of $\mathbb{R}^2$, as desired.

D: $\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda - 2)$, which has eigenvalues $\lambda = 0,2$. This is similar to part C from here.

26. A: Similar idea to problem 24, most of the terms will cancel, giving

$$\begin{pmatrix}1 & 0 & -1\\ 1 & -1 & 0\\ 1 & 1 & -1\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}^{10}\begin{pmatrix}-1 & 1 & 1\\ -1 & 0 & 1\\ -2 & 1 & 1\end{pmatrix}$$

Taking the first few powers of the middle matrix gives

$$\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\to\begin{pmatrix}1 & 2 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\to\begin{pmatrix}1 & 3 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\to\begin{pmatrix}1 & 4 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\ldots$$

It's not hard to confirm this pattern continues indefinitely (you can prove it by induction if you feel fancy). We then just need to compute

$$\begin{aligned}&\begin{pmatrix}1 & 0 & -1\\ 1 & -1 & 0\\ 1 & 1 & -1\end{pmatrix}\begin{pmatrix}1 & 10 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 1 & 1\\ -1 & 0 & 1\\ -2 & 1 & 1\end{pmatrix}\\ &=\begin{pmatrix}1 & 0 & -1\\ 1 & -1 & 0\\ 1 & 1 & -1\end{pmatrix}\begin{pmatrix}-11 & 1 & 11\\ -1 & 0 & 1\\ -2 & 1 & 1\end{pmatrix}\\ &=\begin{pmatrix}-9 & 0 & 10\\ -10 & 1 & 10\\ -10 & 0 & 11\end{pmatrix}\end{aligned}$$

This gives a sum of $\boxed{3}$.

27. D: Similar arguments to earlier problems show that if we let $L = \{\boldsymbol{v}_1 = \langle 1,1,1\rangle, \boldsymbol{v}_2 = \langle 0,-1,1\rangle, \boldsymbol{v}_3\langle -1,0,-1\rangle\}$, then

$$[T_A]_L^L = \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

It follows that $A\boldsymbol{v}_2 = \boldsymbol{v}_1 + \boldsymbol{v}_2$, i.e. $(A-I)\boldsymbol{v}_2 = \boldsymbol{v}_1$, and $A\boldsymbol{v}_1 = \boldsymbol{v}_1$, i.e. $(A-I)\boldsymbol{v}_1 = \langle 0,0,0\rangle$. It thus follows that $(A-I)^2\boldsymbol{v}_2 = \langle 0,0,0\rangle$. Expanding the LHS and rearranging gives $\boxed{(2A-A^2)\boldsymbol{v}_2 = \boldsymbol{v}_2}$.

Remark: $\boldsymbol{v}_2$ is called a generalized eigenvector *of A. There's a lot of super rich theory surrounding generalized eigenvalues and the Jordan canonical form. I encourage you to check it out!*

28. A: The key is to think geometrically about these two transformations. T_A is given to you, and T_B has the effect of cycling the meanings of x, y, and z (it cycles the standard basis vectors). Notably, since all three standard basis vectors are symmetric about $x+y+z=0$, these two operations don't interfere with each other. It follows these operations commute, i.e. $([T_A]B)^{25} = [T_A]^{25}B^{25}$.

Now note that $B^2 = I$ (reflecting over the same plane twice gets you back where you started) and $A^3 = I$ (cycling three times gets you back where you started). Thus $[T_A]^{25}B^{25} = [T_A]B$, so it suffices to compute this. For that, it suffices to compute $[T_A]$. A very similar method to problem 10 works here. To reflect (1,0,0) over $x+y+z=0$, move along a perpendicular vector, i.e. $\langle 1,1,1\rangle$, until you hit $x+y+z=0$, and then move the same amount again. These amount to $\left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$ and $\left(\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$ respectively.

Symmetric calculations and putting the definition together gives

$$[T_A] = \begin{pmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{pmatrix}$$

And then multiplying gives

$$[T_A]B = \begin{pmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \\ 1/3 & -2/3 & -2/3 \end{pmatrix}}$$

29. A: Note that all projections onto lines/planes are linear transformations, so this square will stay a parallelogram. In particular, it suffices to track where the vectors $\langle 1,1\rangle$ and $\langle 1,-1\rangle$ end up, as $\frac{1}{4}$ the area of the original square is $\frac{1}{2}||\langle 1,1\rangle \times \langle 1,-1\rangle||$, a fact preserved through linear transformations (think about the $\frac{1}{2}ab\sin(C)$ formula for the area of a triangle and divide the square into four triangles by drawing in the diagonals). Also note that $\langle 1,-1,0\rangle$ lies on $x+y+nz=0$ for all n, so it is unmoved by the projections. It thus suffices to track where $\langle 1,1,0\rangle$ ends up.

As in the previous problem, projecting onto $x+y+z=0$ involves moving along $\langle 1,1,1\rangle$ until you land on the plane. Since $x+y+z=2$ in our current state, we subtract $\left\langle \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right\rangle$ to end up at $\left\langle \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right\rangle$.

Now we repeat the process. For $x+y+2z=0$ we move along $\langle 1,1,2\rangle$. We have $x+y+2z=-\frac{2}{3}$, so we end up moving to $\left\langle \frac{1}{3}+\frac{1}{9}, \frac{1}{3}+\frac{1}{9}, -\frac{2}{3}+\frac{2}{9}\right\rangle = \left\langle \frac{4}{9}, \frac{4}{9}, -\frac{4}{9}\right\rangle$.

Note that $\left\langle \frac{4}{9}, \frac{4}{9}, -\frac{4}{9}\right\rangle$ is orthogonal to $\langle 1,-1,0\rangle$, so their cross product is just the product of their lengths, or $\frac{4\sqrt{3}}{9} \cdot \sqrt{2} = \frac{4\sqrt{6}}{9}$. Multiplying by $4\left(\frac{1}{2}\right)$ gives $\boxed{\frac{8\sqrt{6}}{9}}$.

30. B: Three are.
I: Yes—$\mathrm{Tr}(A) = -3+1+4 = 2$
II: No—Note that the third row is the sum of the first two, so we can subtract the first two rows from the third and get a row of all 0s, making the determinant 0.
III: Yes—By the previous observation we can zero out the bottom row using Gaussian elimination, but the first two rows are linearly independent, so the rank is 2.
IV: Yes—We can either make a similar observation to the previous two parts (e.g. add five times the first column to three times the third column to zero out a column, and similarly the rank is 2), or invoke the fact that $\mathrm{rank}(A) = \mathrm{rank}(A^T)$ for all matrices A.