Answer Key:

1. **B**
2. **C**
3. **C**
4. **A**
5. **D**
6. **D**
7. **B**
8. **D**
9. **A**
10. **D**
11. **C**
12. **B**
13. **C**
14. **C**
15. **A**
16. **A**
17. **A**
18. **D**
19. **C**
20. **D**
21. **D**
22. **C**
23. **C**
24. **D**
25. **D**
26. **C**
27. **D**
28. **C**
29. **C**
30. **E**

Solutions:

1. $\boxed{\textbf{B}}$: This is the integral $\int_0^1 4799.17 - 4782.26\,dx$. Simply evaluating gives: $4799.17 - 4782.26 = 16.91$

2. $\boxed{\textbf{C}}$: The area of a hexagon is $\frac{3\sqrt{3}}{2}s^2$. Plugging in $s = \sqrt[4]{12}$ gives an area of 9.

3. $\boxed{\textbf{C}}$: The sphere inscribed in the larger cube has diameter 1, which is also the length of the inner cube's space diagonal. If the side length of the cube is s and the volume is s^3, then $s \cdot \sqrt{3} = 1$ gives $s^3 = \sqrt{\frac{1}{27}}$. Therefore, the final answer is $27 + 1 = 28$

4. $\boxed{\textbf{A}}$: Since it is a right triangle, we can fix one of the angles to be right and construct our triangle. Noticing $\theta_2 = \frac{\pi}{2}$, we get a triangle similar to $5 - 12 - 13$ right triangle, which has perimeter 30. So, this triangle of perimeter 10 will have each side length be a third of its corresponding side, giving a final area of $\frac{5}{3} \cdot \frac{12}{3} \cdot \frac{1}{2} = \frac{10}{3}$.

5. $\boxed{\textbf{D}}$: This is an ellipse with foci at $(0, 8i)$ and $(6, 0)$ The distance between the foci is 10 and the sum of the distances to each foci is 16, giving $a = 13$, $c = 5$, and $b = \sqrt{a^2 - c^2} = 12$. So, the area is simply $\pi \cdot a \cdot b = 156\pi$.

6. $\boxed{\textbf{D}}$: The length of the latus rectum of an ellipse is $\frac{2b^2}{a}$. Plugging in $a = 13$ and $b = 12$ gives a length of $\frac{288}{13}$, which go through each of the foci. So, the hexagon is comprised of a rectangle with area $\frac{288}{13} \cdot 10$ and two triangles, each with height $a - c = 13 - 5 = 8$ and area $8 \cdot \frac{288}{13} \cdot \frac{1}{2}$. So, the total area is $Rectangle + 2 \cdot Triangle = (10 + 8) \cdot \frac{288}{13} = \frac{5184}{13}$.

7. $\boxed{\textbf{B}}$: An ellipsoid has volume $\frac{4\pi abc}{3}$, where a, b, c are the lengths of the 3 semi-axes. Since $a = 13$ and $b = 12$ from out initial calculations, we only need to find the third semi-axis. Revolving an ellipse around its major axis means the third semi-axis is also the minor semi-axis, giving $c = 12$. Therefore, the total volume is $\frac{4\pi \cdot 12 \cdot 12 \cdot 13}{3} = 2496\pi$

8. $\boxed{\textbf{D}}$: We can use Pappus' Theorem to find the volume of each funnel cake, which says:

$$V = 2\pi \cdot A \cdot d$$

where A is the area of the shape and d is the distance from which the centroid is revolving. Since each circle has area of $(\frac{1}{5})^{2n} \cdot \pi$ and is being revolved by a distance of 2^n, we have each volume is $V_n = 2\pi \cdot \pi \cdot (\frac{2}{25})^n = 2\pi^2(\frac{2}{25})^n$ from $n = 0$ to $n = \infty$. This gives a sum of $\frac{2\pi^2}{1 - \frac{2}{25}} = \frac{50\pi^2}{23}$, or a compiled answer of 75.

9. $\boxed{\textbf{A}}$: Note that $|x|$ represents a piece-wise function, where $|x| = x$ for $x >= 0$, and $|x| = -x$ for $x < 0$. Therefore, we can look at this region quadrant by quadrant, seeing how it changes as both x and y switch from positive to negative. Therefore, each quadrant will represent a different line segment connecting into a shape. The equations are:

$$Q_1 : 12x + 6y = 36$$
$$Q_2 : -6x + 6y = 36$$
$$Q_3 : -6x - 2y = 36$$
$$Q_4 : 12x - 2y = 36$$

This defines a quadrilateral made up of 4 line connecting line segments that enclose a triangle in their respective quadrants. To find the area of the quadrilateral, simply find the sum of the areas of each triangle, which is just the magnitude of each line segment's x-intercept times its y-intercept divided by two. The final area is:

$$\frac{3 \cdot 6}{2} + \frac{6 \cdot 6}{2} + \frac{6 \cdot 18}{2} + \frac{3 \cdot 18}{2} = 108$$

10. $\boxed{\text{D}}$: We notice the two functions intersect at $x = \frac{\sqrt{2}}{2}$ and $x = \frac{-\sqrt{2}}{2}$. We set up the integral as follows:

$$A = \int_a^b f(x) - g(x)\,dx$$

$$A = \int_{\frac{-\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} (1 - x^2) - (x^2)\,dx = 2\int_0^{\frac{\sqrt{2}}{2}} (1 - 2x^2)$$

$$A = 2(x - \frac{2x^3}{3})\Big|_0^{\frac{\sqrt{2}}{2}} = \frac{2\sqrt{2}}{3}$$

11. $\boxed{\text{C}}$: Since n is a scalar, we can take it out of the integral to evaluate $\lim\limits_{n\to\infty} n\int_0^1 x^n\,dx$. The integral inside evaluates to $\frac{1}{n+1}$, giving a value of $\frac{n}{n+1}$. As n approaches infinity, we get a limit of 1.

12. $\boxed{\text{B}}$: This function can be expressed as $(x-1)^6 + x^6$. So, evaluating the integral $\int_1^2 (x-1)^6 + x^6\,dx$, we can split the integral to get $\int_1^2 (x-1)^6\,dx + \int_1^2 x^6\,dx$. Using $u = x - 1$ evaluates the first integral to $\frac{1}{7}$ and the second integral comes out to $\frac{127}{7}$. This gives a final answer of $\frac{128}{7}$.

13. $\boxed{\text{C}}$: The area of the region can be found by evaluating the following integral:

$$\int k^x\,dx = \frac{k^x}{\ln k}$$

So, evaluating gives $\int_0^1 k^x\,dx = \frac{k-1}{\ln k}$ and $\int_1^2 k^x\,dx = \frac{k^2-k}{\ln k} = k \cdot \frac{k-1}{\ln k}$. The ratio of the two areas $\frac{k\cdot\frac{k-1}{\ln k}}{\frac{k-1}{\ln k}} = k$. So, $\sqrt{17} : \sqrt{6} = 1 : k$, giving $k = \sqrt{\frac{6}{17}}$ and $k^2 = \frac{6}{17}$, and a final compiled answer of $6 + 17 = 23$.

14. $\boxed{\text{C}}$: Drawing each cardioid, we notice the intersection is four congruent areas, $r = 1 + \cos\theta$ from $\theta = \frac{\pi}{2}$ to $\theta = \pi$, meaning we only need to integrate once and multiply by 4 for the total area. Following the polar area formula of $\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta$, we plug in values to get the following expression for the area:

$$4 \cdot \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (1 + \cos\theta)^2\,d\theta = 2\int_{\frac{\pi}{2}}^{\pi} \cos^2\theta + 2\cos\theta + 1$$

Rewriting with $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$, we have $\int_{\frac{\pi}{2}}^{\pi} \cos(2\theta) + 4\cos\theta + 3\,d\theta = 0 - 4 + \frac{3\pi}{2} = \frac{3\pi - 8}{4}$

15. $\boxed{\text{A}}$: We can use the substitution $x = r\cos\theta, y = r\sin\theta$ to transform this nonlinear equation into polar coordinates. Doing so gives $x^2 + y^2 = r^2$ and $x^2y^2 = r^4\cos^2\theta\sin^2\theta$, giving:

$$r^8 = 8r^4\cos^2\theta\sin^2\theta$$

Dividing by r^4 and using $\sin(2\theta) = 2\sin\theta\cos\theta$, this simplifies to $r^4 = 2\cdot\sin^2(2\theta)$. Finally, we can take the square root to get $r^2 = \sqrt{2}\sin(2\theta)$, which can be used in the equation for finding polar area as given by $\frac{1}{2}\int r^2\,d\theta$. Since x, y are real numbers, we can substitute to get:

$$\frac{1}{2}\int_0^{2\pi} \sqrt{2}\sin(2\theta)\,d\theta = \frac{1}{2}\cdot\sqrt{2}\cdot 4\int_0^{\frac{\pi}{2}} \sin(2\theta)\,d\theta$$

Evaluating the integral gives a value of $\frac{1}{2}$, making the product $\frac{1}{2}\cdot\sqrt{2}\cdot 4\cdot\frac{1}{2} = \sqrt{2}$.

16. **A** : Let two points A and B be chosen uniformly at random on the unit circle centered at the origin, and let the third point $O = (0, 0)$ be fixed. We parametrize the points as the following, knowing θ_1, θ_2 are distributed uniformly on the interval $[0, 2\pi]$:

$$A = (\cos\theta_1, \sin\theta_1), \quad B = (\cos\theta_2, \sin\theta_2)$$

The area of the is given by the half sine area formula, knowing two of the side lengths are the radii of the unit circle:

$$A = \frac{1}{2}|\sin(\theta_2 - \theta_1)|$$

Letting $t = \theta_2 - \theta_1 \mod 2\pi$, the expected area becomes:

$$\mathbb{E}[A] = \frac{1}{2} \cdot \mathbb{E}|\sin(t)| = \frac{1}{2} \cdot \frac{1}{2\pi} \int_0^{2\pi} |\sin t| \, d\phi$$

We evaluate the integral:

$$\int_0^{2\pi} |\sin t| \, dt = 4$$

So the expected area is:

$$\frac{1}{2} \cdot \frac{1}{2\pi} \cdot 4 = \frac{1}{\pi}$$

17. **A** : Let two points A and B be chosen uniformly at random on the unit circle centered at the origin, and let the third point $O = (0, 0)$ be fixed. We parametrize the points as:

$$A = (\cos\theta_1, \sin\theta_1), \quad B = (\cos\theta_2, \sin\theta_2)$$

where $\theta_1, \theta_2 \sim \text{Uniform}[0, 2\pi]$. Since $OA = OB = 1$, and the chord length $AB = 2\sin\left(\frac{t}{2}\right)$ where $t = |\theta_2 - \theta_1| \mod 2\pi$, the perimeter of triangle $\triangle OAB$ is:

$$P = 1 + 1 + 2\sin\left(\frac{t}{2}\right) = 2 + 2\sin\left(\frac{t}{2}\right)$$

To compute the expected perimeter, we use the fact that $t \sim \text{Uniform}[0, 2\pi]$, so:

$$\mathbb{E}[P] = 2 + 2 \cdot \mathbb{E}\left[\sin\left(\frac{t}{2}\right)\right]$$

Change variables with $u = \frac{t}{2} \Rightarrow t = 2u,\ dt = 2\,du$, and when $t \in [0, 2\pi]$, $u \in [0, \pi]$:

$$\mathbb{E}\left[\sin\left(\frac{t}{2}\right)\right] = \frac{1}{2\pi}\int_0^{2\pi} \sin\left(\frac{t}{2}\right) dt = \frac{1}{2\pi} \cdot 2\int_0^{\pi} \sin u \, du = \frac{2}{\pi}$$

Thus, the expected perimeter is:

$$\mathbb{E}[P] = 2 + 2 \cdot \frac{2}{\pi} = \frac{2\pi + 4}{\pi}$$

18. **D** : Imagining the image on the Cartesian plane, we can position the image such that the leftmost square has vertices $(0, 0), (0, 1), (1, 0), (1, 1)$. Now, assigning the points as $(X_1, Y_1), (X_2, Y_2), (X_3, Y_3)$ from left to right squares, we can find the area of the triangle through shoelace as:

$$\frac{1}{2}|X_1Y_2 + X_2Y_3 + X_3Y_1 - Y_1X_2 - Y_2X_3 - Y_3X_1|$$

The expected area of the triangle is simply the expected area of this function. Since the points are randomly chosen in a square such that each X_i, Y_i is independent, we can simply find the expected value for each X_i, Y_i and evaluate the product and sum. From uniform distributions, we get $E(X_1) = E(Y_1) = E(Y_3) = \frac{1}{2}$, $E(X_2) = E(Y_2) = \frac{3}{2}$, and $E(X_3) = \frac{5}{2}$. Plugging in these values gives:

$$\frac{1}{2} \cdot |\frac{1}{2} \cdot \frac{3}{2} + \frac{3}{2} \cdot \frac{1}{2} + \frac{5}{2} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{3}{2} - \frac{3}{2} \cdot \frac{5}{2} - \frac{1}{2} \cdot \frac{1}{2}| = 1$$

So, we get the expected value for the area is 1.

19. **C** : Take a general parabola of the form $f(x) = \frac{1}{4a}x^2$, where a is the focal distance. Since the length of the latus rectum is $4a$, the area between the parabola and its latus rectum is:

$$f(a) = \int_{-2a}^{2a} a - \frac{1}{4a}x^2 \, dx = 2(ax - \frac{x^3}{12a})\Big|_0^{2a}$$

Evaluating gives $f(a) = \frac{8a^2}{3}$. Now, we can evaluate the integral and get our final answer:

$$\int_0^3 \frac{8a^2}{3} \, da = \frac{8a^3}{9}\Big|_0^3 = 24$$

20. **D** : Consider that this shape has 6 ”sides”. The bottom face has area of $1^2 = 1$. Next, consider the top face. Since all of the cubes are stacked directly on top of the original cube, the total surface area here is a telescoping series evaluating to $1 - \frac{1}{2}^2 + \frac{1}{2}^2 - \frac{1}{3}^2 + ... = 1$. Lastly, there are four lateral faces, each with a total surface area of $1 + (\frac{1}{2})^2 + (\frac{1}{3})^2 + ... = \frac{\pi^2}{6}$. So, the final surface area is:

$$1 + 1 + 4 \cdot \frac{\pi^2}{6} = \frac{6 + 2\pi^2}{3}$$

21. **D** : This can be rewritten as a double integral:

$$\int_0^\infty \int_0^\infty e^{\frac{-(x+y)}{\sqrt{5}}} \, dx \, dy$$

Using separation, we get:

$$\int_0^\infty e^{\frac{-x}{\sqrt{5}}} \, dx \int_0^\infty e^{\frac{-y}{\sqrt{5}}} \, dy$$

Each integral evaluates to $e^{\frac{-x}{\sqrt{5}}}\Big|_0^\infty = \sqrt{5}$, giving a final answer of $\sqrt{5} \cdot \sqrt{5} = 5$

22. **C** : Let us construct two cylinders, $y^2 + z^2 = r^2$ and $x^2 + y^2 = r^2$. We can look at the cross sections of the intersections of the two cylinders and notice we have $y \in [-\sqrt{r^2 - z^2}, \sqrt{r^2 - z^2}]$ and $x \in [-\sqrt{r^2 - z^2}, \sqrt{r^2 - z^2}]$, making the cross section a square with side length $2\sqrt{r^2 - z^2}$, and cross-sectional area $4(r^2 - z^2)$.

Since z varies from $-r$ to r, we can simply integrate:

$$A = 4\int_{-r}^{r} r^2 - z^2 \, dz = 4(r^2 z - \frac{z^3}{3})\Big|_{-r}^{r} = \frac{16r^3}{3}$$

Alternatively, we can notice these cross sections are equivalent to a square inscribed in a circle where the ratio of the areas is $\frac{4}{\pi}$, meaning we can just multiple the area of the volume of a sphere to get $\frac{4\pi r^3}{3} \cdot \frac{4}{\pi} = \frac{16r^3}{3}$. Plugging in $r = 2$ gives a final value of $\frac{128}{3}$.

23. **C** : We can take one half of the hexagon after it has been split by its longest diagonal, and segment it into two congruent triangles and a rectangle. When these shapes are revolved along the main diagonal, it results in a cylinder and two congruent cones above and below it.

The height of the cylinder is the side length of 4. The radius of the cylinder and both cones is just the hexagon's apothem, $\frac{\sqrt{3}}{4}s = 2\sqrt{3}$ where s is the side length of 4. Lastly, the sum of the heights of the two cones plus the cylinder height is the length of the main diagonal, which is $2 \cdot 4 = 8$. So, the height of each cone is 2. We can now plug in each value and solve to get:

$$V = 2V(\text{Cone}) + V(\text{Cylinder}) = 2 \cdot \frac{\pi}{3} \cdot (2\sqrt{3})^2 \cdot 2 + \pi \cdot (2\sqrt{3})^2 \cdot 4 = 64\pi$$

24. **D** : By shell method, this volume is the integral:

$$\int_0^\infty 2\pi \cdot x \cdot (\frac{\sin x}{x^2})\, dx = 2\pi \int_0^\infty (\frac{\sin x}{x})\, dx$$

We define a similar version of the integral using a parameter $a > 0$, which allows absolute convergence and allows us to differentiate under the integral sign:

$$I(a) = \int_0^\infty \frac{\sin x}{x} e^{-ax}\, dx$$

We use the identity $\frac{\sin x}{x} = \int_0^1 \cos(tx)\, dt$ to rewrite the integral:

$$I(a) = \int_0^\infty \left(\int_0^1 \cos(tx)\, dt \right) e^{-ax}\, dx$$

By Fubini's Theorem, we can switch the order of integration:

$$I(a) = \int_0^1 \left(\int_0^\infty \cos(tx) e^{-ax}\, dx \right) dt$$

Evaluating the inner integral using Laplace transform identities:

$$\int_0^\infty \cos(tx) e^{-ax}\, dx = \frac{a}{a^2 + t^2}$$

So we plug this back into our expression for $I(a)$:

$$I(a) = \int_0^1 \frac{a}{a^2 + t^2}\, dt = \tan^{-1}\left(\frac{1}{a}\right)$$

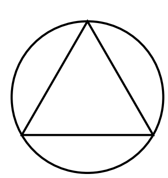
Taking the limit as $a \to 0^+$, we obtain the integral:

$$\int_0^\infty \frac{\sin x}{x}\, dx = \lim_{a \to 0^+} I(a) = \lim_{a \to 0^+} \tan^{-1}\left(\frac{1}{a}\right)$$

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

So, the integral evaluated from 0 to ∞ is exactly $\frac{\pi}{2}$. We can now multiply by 2π to get a final answer of π^2.

25. **D** : We can take a drawing of the triangle and the sphere's great circle to be:



Since the sphere has a radius of 1, we can construct a line segment from the circle to its center, creating a perpendicular bisector to one of the triangle's sides. Since the apothem has length $\frac{1}{2}$, the height of each spherical cap is the radius of the sphere minus the apothem, giving $1 - \frac{1}{2} = \frac{1}{2}$. We can now use the formula for the volume of a spherical cap derived earlier:

$$V = \frac{\pi h^2(3R - h)}{3}$$

Using the radius R of the sphere as 1 and the height h of the spherical cap as $\frac{1}{2}$, we get the volume of each cap as $\frac{5\pi}{24}$, giving a compiled answer of $5 + 24 = 29$.

26. **C** : Rather than actually integrating over some strange regions, we can just use the fact that the volume of the sphere is 3 times the volume of each Goober plus the volume of the Goofball. The volume of the sphere is $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi$ when $r = 1$. So, we can solve to get:

$$\frac{4}{3}\pi - 3 \cdot \frac{5\pi}{24} = \frac{17\pi}{24}$$

So, our final compiled answer is $17 + 24 = 41$.

27. **D** : Since this is a spherical cap created by slicing the sphere into 8 congruent pieces through a heptagonal shaped wire, we need to first find the surface area of a spherical cap. We derive the surface area of a spherical cap using the formula for surface area. Consider the upper hemisphere of a sphere of radius R, given by:

$$y = \sqrt{R^2 - x^2}$$

We revolve this curve about the x-axis from $x = -a$ to $x = a$, where a is the radius of the circular base of the cap. The height of the cap is $h = R - \sqrt{R^2 - a^2}$.

The surface area of revolution is given by:

$$A = 2\pi \int_{-a}^{a} y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Now plug into the surface area formula:

$$A = 2\pi \int_{-a}^{a} \sqrt{R^2 - x^2} \cdot \sqrt{\frac{R^2}{R^2 - x^2}}\, dx = 2\pi \int_{-a}^{a} \sqrt{R^2}\, dx = 2\pi R \int_{-a}^{a} dx$$

$$A = 2\pi R(2a) = 4\pi Ra$$

But this is the full half-sphere's lateral area when $a = R$. For the sliced portion, we integrate only from $x = -a$ to $x = a$, which is already done. Now we relate a and h using the right triangle:

$$a^2 = R^2 - (R - h)^2 = 2Rh - h^2$$

So:

$$A = 2\pi Rh = \pi(a^2 + h^2)$$

Thus, the total surface area of the spherical cap, including the base of area πa^2 is:

$$A = \pi(2a^2 + h^2)$$

We know that the height is the radius of the sphere, 1, minus the length of the apothem of the polygon, given by $\cos(\frac{\pi}{n})$, where n is the number of sides. The radius of the cap a can be found by creating a right triangle with height $\cos(\frac{\pi}{n})$ and hypotenuse 1 giving $a = \sin(\frac{\pi}{n})$. So, plugging in the values for a and h gives:

$$S_1(n) = \pi(2a^2 + h^2) = \pi(2\sin(\frac{\pi}{n})^2 + (1 - \cos(\frac{\pi}{n})^2)$$

Expanding and simplifying with the fact that $\sin^2 + \cos^2 = 1$

$$S_1(n) = \pi(2(1-\cos(\frac{\pi}{n})^2) + \cos(\frac{\pi}{n})^2 + 1 - 2\cos(\frac{\pi}{n}) = \pi(3 - 2\cos(\frac{\pi}{n}) - \cos(\frac{\pi}{n})^2)$$

This gives $S_1(x) = \pi(3 - 2x - x^2)$. Plugging in $x = \frac{1}{3}$ gives a value of $\frac{20\pi}{9}$.

28. $\boxed{\textbf{C}}$: Since the two points chosen will always create a chord, we can generalize this problem. Take the unit circle on the standard Cartesian plane given by $x^2 + y^2 = 1$. Then, we can fix one point on $(1, 0)$ and the other somewhere on the unit circle such that the angle between them is θ. From symmetry, $0 \le \theta \le \pi$ and θ is uniform since the points are chosen at random.

Now, we just need to find a function $f(\theta)$ that gives us the smaller area created by the constructed chord, so we can then find the average value. We can imagine the smaller area as a sector minus a triangle: the area of a sector of a unit circle is $\frac{1}{2}r^2\theta$, and the area of the triangle is $\frac{1}{2}r^2\sin\theta$. So, when $r = 1$, $f(\theta) = \frac{\theta - \sin\theta}{2}$. We can now integrate to find the average area, A:

$$A = \frac{\int_a^b f(x)\,dx}{b-a} = \frac{\int_0^\pi \frac{\theta - \sin\theta}{2}}{\pi - 0} = \frac{1}{2\pi}\left(\frac{\theta^2}{2} + \cos\theta)\right|_0^\pi = \frac{\pi^2 - 4}{4\pi}$$

29. $\boxed{\textbf{C}}$: There is a $\frac{4}{5}$ chance of the coin landing on either of its faces, both with area π, so the total surface area of the coin is $\frac{5}{4} \cdot 2\pi = \frac{5\pi}{2}$. Thus, the area of the side is $\frac{\pi}{2}$. The circumference of the coin is 2π, so the thickness of the coin is $\frac{\pi/2}{2\pi} = \frac{1}{4}$.

30. $\boxed{\textbf{E}}$: The semiperimeter of the triangle is 21. Using Heron's formula, the area is $\sqrt{21(21-13)(21-14)(21-15)} = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$. The greatest integer less than this is 83.