Answer Key:

1. **E**
2. **A**
3. **B**
4. **B**
5. **A**

6. **A**
7. **C**
8. **A**
9. **D**
10. **D**

11. **C**
12. **B**
13. **D**
14. **B**
15. **D**

16. **B**
17. **A**
18. **C**
19. **C**
20. **D**

21. **C**
22. **B**
23. **C**
24. **A**
25. **B**

26. **B**
27. **A**
28. **B**
29. **A**
30. **C**

Solutions:

1. $\boxed{\text{E}}$: $x^{3/2}$ is undefined for $x < 0$, so $\lim\limits_{x\to 0}\frac{x^{3/2}}{\sin x}$ does not exist.

2. $\boxed{\text{A}}$: We can use implicit differentiation to solve for $\frac{dx}{dy}$:

$$x^2 - 3y + 6xy + 9y^2 - x = 6$$

$$2x\left(\frac{dx}{dy}\right) - 3 + 6x + 6y\left(\frac{dx}{dy}\right) + 18y - \frac{dx}{dy} = 0$$

   At the given point $(-3,2)$, $\frac{dx}{dy} = \boxed{-3}$

   Alternatively, notice that if we let $u = x + 3y$, the given equation is equal to $u^2 - u - 6 = 0$. Taking the derivative with respect to y gives $(2u-1)\left(\frac{du}{dy}\right) = (2u-1)\left(\frac{dx}{dy}+3\right) = 0 \Rightarrow \frac{dx}{dy} = -3$.

3. $\boxed{\text{B}}$: Plugging in $x = 3$, we obtain the fraction $\frac{0}{2} = \boxed{0}$.

4. $\boxed{\text{B}}$: Since $f(x) = x^3 + 5x^2 + 8x + 2$ is continuous on the interval $[-3,-1]$ and differentiable on the interval $(-3,-1)$, the MVT states that there exists some value $x = c$ in the interval $(-3,-1)$ such that

$$\frac{f(-1)-f(-3)}{(-1)-(-3)} = f'(c)$$

   $f(-3) = -4$ and $f(-1) = -2$, and solving $3c^2 + 10c + 8 = 1$, we obtain the values $c = -\frac{7}{3}$ and $c = -1$.

   The latter does not work because it is an endpoint of the interval; thus, the sum of values is $\boxed{-\frac{7}{3}}$

5. $\boxed{\text{A}}$: First expand the equation:

$$D = \sin\left(x - \frac{\pi}{6}\right) + \cos x = \frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x + \cos x = \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x$$

   Then, take the derivative and solve for x:

$$D' = \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = 0 \Rightarrow \tan x = \sqrt{3} \Rightarrow x = \frac{\pi}{3}, \frac{4\pi}{3}$$

   Plugging back these two values, we see that the maximum value of D is $\boxed{1}$
   (Alternatively, the amplitude can be found with $\sqrt{(\frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2} = 1$.)

6. $\boxed{\text{A}}$: Compare the largest limits on the numerator and denominator. As $n \to \infty$, $n^2 \ln n < n^3$, so $3n^3$ is the largest term in the numerator; similarly, $6n^3$ is the largest term in the denominator because it has the highest power. $\lim\limits_{x\to\infty}\frac{3n^3}{6n^3} = \boxed{\frac{1}{2}}$

7. $\boxed{\text{C}}$: $f(x) = x\sin^3 x \Rightarrow f'(x) = \sin^3 x + 3x\sin^2 x\cos x \Rightarrow f'\left(\frac{\pi}{6}\right) = \frac{2+\pi\sqrt{3}}{16} \Rightarrow a+b+c = \boxed{21}$.

8. $\boxed{\text{A}}$: $r = \sqrt{8+8\cos\theta} \Rightarrow r' = \frac{-8\sin\theta}{2\sqrt{8+8\cos\theta}}$. When $\theta = \frac{\pi}{3}$, $r = 2\sqrt{3}$, $r' = -1$, $\sin\theta = \frac{\sqrt{3}}{2}$, and $\cos\theta = \frac{1}{2}$. Plugging in these values, $\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \boxed{-\frac{\sqrt{3}}{7}}$.

9. $\boxed{\textbf{D}}$: Let $y_n = x_n^2 - 6x_n + 2$, $m_n = \frac{dy}{dx} = 2x_n - 6$, and $x_{n+1} = x_n - \frac{y_n}{m_n}$. We can set up a table as follows:

n	x_n	y_n	m_n	x_{n+1}
0	1	-3	-4	$1/4$
1	$1/4$	$9/16$	$-11/2$	$31/88$
2	$31/88$			

$x_2 = \boxed{\frac{31}{88}}$

10. $\boxed{\textbf{D}}$: The area of each petal of the rose curve $r = a\sin(n\theta)$ is $\int_0^{\frac{\pi}{n}} \frac{a^2}{2}\sin^2(n\theta)d\theta$, so the total area of a rose curve with an odd number of petals (as with $n = 3$) is $\frac{\pi a^2}{4}$.

$A = \frac{\pi a^2}{4} \Rightarrow \frac{dA}{dt} = \frac{a\pi}{2} \cdot \frac{da}{dt}$. Since $a' = 3t^2$, $a = t^3 + C$; since $a = 3$ when $t = 1$, $C = 2$. Plugging these equations back in, we get that $\frac{dA}{dt} = \frac{a\pi}{2}(3t^2)(t^3+2)$, which equals $\boxed{60\pi}$ when $t = 2$.

11. $\boxed{\textbf{C}}$: $f(1) = \left(x^3 + 3x^2 + \frac{4}{x}\right)\Big|_{x=1} = 8$, and $f'(1) = \left(3x^2 + 6x - \frac{4}{x^2}\right)\Big|_{x=1} = 5$. Plugging in $x = 1.01$ into the tangent line $y = 5(x-1)+8$ gives $y = \boxed{8.05}$

12. $\boxed{\textbf{B}}$: Recall the Taylor series representations of the following functions:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

We can manipulate this first series to obtain:

$$xe^{x^2} = x + x^3 + \frac{x^5}{2!} + \cdots$$

Plugging these in, our desired limit Find $\lim_{x\to 0} \frac{xe^{x^2} - \tan x}{\sin x - x} = \frac{(x + x^3 + \cdots) - (x + \frac{x^3}{3} + \cdots)}{(x - \frac{x^3}{6} + \cdots) - x} = \frac{2/3}{-1/6} = \boxed{-4}$

13. $\boxed{\textbf{D}}$: Let $y = x^{\ln x + 1}$. Then,

$$\ln y = \ln x(\ln x + 1) = \ln^2 x + \ln x$$

$$\frac{y'}{y} = \frac{2\ln x + 1}{x}$$

At $x = e$, $y = e^2$, so $y' = e^2\left(\frac{3}{e}\right) = \boxed{3e}$

14. $\boxed{\textbf{B}}$: Using Leibniz's Rule for derivatives, $(uv)^{(n)} = \binom{n}{0}(u)^{(n)} + \binom{n}{1}(u)^{(n-1)}(v)^{(1)} + \binom{n}{2}(u)^{(n-2)}(v)^{(2)} + \cdots$ Therefore, $f^{(2025)}(x^3e^x) = \binom{2025}{0}e^x x^3 + \binom{2025}{1}e^x(3x^2) + \binom{2025}{2}e^x(6x) + \binom{2025}{3}e^x(6)$ and $f^{(2025)}(0) = 6\binom{2025}{3} =$ $\boxed{2025 * 2024 * 2023}$

15. $\boxed{\text{D}}$: Recall that the Taylor Series of $f(x)$ centered about $x=c$ can be expressed as $\sum\limits_{n=0}^{\infty}(\frac{f^{(n)}(c)(x-c)^n}{n!})$ and that $e^x=\sum\limits_{n=0}^{\infty}(\frac{x^n}{n!})$. Now, substitute $u=x^3$ and multiply each term by x^3 (to obtain the Maclaurin series of $x^3e^{x^3}$). This is then equivalent to $\sum\limits_{n=0}^{\infty}(\frac{x^{3n+3}}{n!})$. Taking the term with x^{2025}, we can see that $\frac{(g^{(2025)}(0))x^{2025}}{2025!}=\frac{x^{2025}}{674!}$. Thus, $f^{(2025)}(0)=\boxed{\frac{2025!}{674!}}$ (Note that this method can also be used to solve #4.)

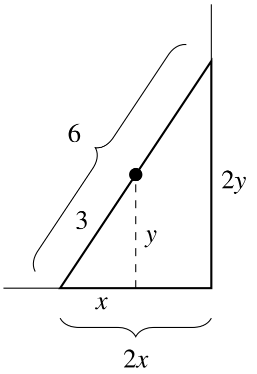
16. $\boxed{\text{B}}$: Dividing both sides of $(y'\cos x+y\sin x=x\cos^4 x)$ by $\cos^2 x$ gives $y'\sec x+y\sec x\tan x=x\cos^2 x$. The left side is the derivative of $(y\sec x)$ with respect to x, and, integrating, we get

$$y\sec x=\int x\cos^2 x=\int\frac{x+x\cos(2x)}{2}=\frac{x^2}{4}+\frac{x}{4}\sin(2x)+\frac{1}{8}\cos(2x)+C$$

Plugging in the point $(0,0)$ gives $C=-\frac{1}{8}$; then, when $x=\frac{\pi}{4}$, $y=\frac{\sqrt{2}}{128}(\pi^2+4\pi-8)$, which gives $a+b+c=\boxed{124}$.

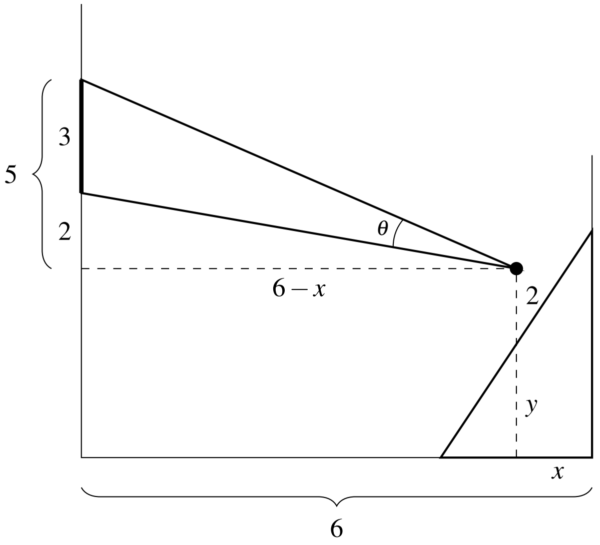
17. $\boxed{\text{A}}$: Let $L=\lim\limits_{x\to\infty}\sqrt[3]{x^6+2x^4-1}-\sqrt[4]{x^8+4x^6+2x^2+2}=\lim\limits_{x\to\infty}(x^6+2x^4)^{1/3}-(x^8+4x^6)^{1/4}$ $=\lim\limits_{x\to\infty}\frac{(1+2/x^2)^{1/3}-(1+4/x^2)^{1/4}}{1/x^2}$. Substituting $u=1/x^2$ and using L'Hopital's rule, $L=\lim\limits_{u\to0}\frac{(1+2u)^{1/3}-(1+4u)^{1/4}}{u}\Rightarrow$ $\lim\limits_{x\to0}\frac{2}{3}(2u+1)^{-2/3}-(4u+1)^{-3/4}=\frac{2}{3}-1=\boxed{-\frac{1}{3}}$

18. $\boxed{\text{C}}$: We draw the following diagram, where Tiger is standing at the midpoint of the hypotenuse of the triangle:



Then, using the Pythagorean Theorem, $x^2+y^2=9\Rightarrow 2x(x')+2y(y')=0$
$2x=2$ and $2(y')=-2\sqrt{2}\Rightarrow x=1$, $y=2\sqrt{2}$ and $y'=-\sqrt{2}$. Plugging in our values, we find that $x'=\boxed{4}$

19. $\boxed{\text{C}}$: Adding the second wall to the previous problem, we get the following diagram: (Note that the dot now corresponds to the height of Tiger's eyes above the ground rather than the height of his feet.)



Using the tangent subtraction formula with $\tan(\alpha) = \frac{5}{6-x}$ and $\tan(\beta) = \frac{2}{6-x}$, $\tan(\theta) = \frac{\frac{3}{6-x}}{1+\frac{10}{(6-x)^2}} = \frac{3(6-x)}{(6-x)^2+10} = \frac{-3(x-6)}{x^2-12x+46}$. Then, taking the derivative, $\sec^2(\theta)(\theta') = \frac{-3(x^2-12x+46)+6(x-6)^2}{(x^2-12x+46)^2}(x')$. $\sec^2(\theta) = 1+\tan^2(\theta)$, and plugging in $x = 1$ and $x' = 4$ (found in the previous problem), we find that $\tan(\theta) = \frac{15}{35}$, $\sec^2(\theta) = \frac{1450}{1225}$, and $\theta' = 4(\frac{45}{1225})(\frac{1225}{1450}) = \frac{18}{145} \Rightarrow a+b = \boxed{163}$.

20. $\boxed{\textbf{D}}$: By L'Hopital's rule,

$$\lim_{h\to 0}\frac{f(3h-1)-f(-h-1)}{2h} \Rightarrow \lim_{h\to 0}\frac{3f'(3h-1)+f'(-h-1)}{2} = 2f'(-1)$$

$f(x) = x^2+3x+4 \Rightarrow f'(-1) = (2x+3)\Big|_{x=-1} = 1$, so $2f'(-1) = \boxed{2}$

21. $\boxed{\textbf{C}}$: By the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = u'(x)f(u(x)) - v'(x)f(v(x))$$

In this case, the upper bound is a constant, but the lower bound is not, so

$$f'(x) = \frac{d}{dx}\int_{x^2}^{2} t^2\log_3(t)dt = -2x(x^4)\log_3(x^2)$$

Therefore, $f'(\sqrt{3}) = \boxed{-18\sqrt{3}}$

22. $\boxed{\textbf{B}}$:

$$f(x) = \arctan\left(\frac{2}{3-x}\right) \Rightarrow f'(x) = \frac{2}{(3-x)^2+4} \Rightarrow \lim_{x\to 3} f'(x) = \boxed{\frac{1}{2}}$$

(In this case, the discontinuity in $f(x)$ does not affect the limit of the derivative.)

23. $\boxed{\textbf{C}}$: $f'(x) = 2x^3+4x^2+2x+1$, and $f''(x) = 6x^2+8x+2$. To find the critical points of $f'(x)$, we let $f''(x) = 0$, which gives $x = -1$ and $x = -\frac{1}{3}$. Plugging these back into $f'(x)$, we get 1 and $\frac{19}{27}$, respectively, and the positive difference between the two is $\boxed{\frac{8}{27}}$

24. $\boxed{\textbf{A}}$: Substituting $m = 2n$ gives

$$L = \lim_{n\to\infty}\left(\sum_{i=2}^{m}\frac{\frac{\pi}{2}}{m(1+\tan^4(\frac{\pi i}{2m}))}\right)$$

and using the limit (Riemann sum) definition of an integral,

$$L = \int_0^{\frac{\pi}{2}}\frac{1}{1+\tan^4 x}dx$$

Substituting $u = \frac{\pi}{2} - x$ then gives

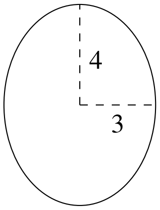
$$L = \int_0^{\frac{\pi}{2}}\frac{1}{1+\cot^4 x}dx = \int_0^{\frac{\pi}{2}}\frac{\tan^4 x}{\tan^4 x+1}dx$$

since $\cot(x) = \dfrac{1}{\tan x} = \tan(\dfrac{\pi}{2} - x)$. Then, we add up the two equal versions of L, giving

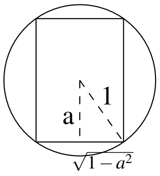
$$2L = \int_0^{\frac{\pi}{2}}\frac{\tan^4 x+1}{\tan^4 x+1} = \frac{\pi}{2}$$

Therefore, $L = \boxed{\dfrac{\pi}{4}}$.

25. $\boxed{\textbf{B}}$: The curve of $y = \pm\frac{4}{3}\sqrt{9-x^2}$ is equivalent to that of $\frac{x^2}{9} + \frac{y^2}{16} = 1$. Now take a cross section from a plane along the major axis of this ellipse:



Let's now simplify this problem to a circle with radius 1 (this is not necessary but is slightly simpler):



Then, the radius of this cylinder is $V = \pi(\sqrt{1-a^2})^2(2a) = 2\pi(a-a^3)$. To maximize V, let $V' = 2\pi(1-3a^2) = 0 \Rightarrow a = \frac{1}{\sqrt{3}} \Rightarrow V = \frac{4\pi\sqrt{3}}{9}$. Then, scaling this back up to the cylinder in the ellipsoid, we multiply the volume by the "radii" in the three directions corresponding to the axes. These lengths correspond to 4, 3, and 3, respectively, so $V = \frac{4\pi\sqrt{3}}{9} \cdot 9 \cdot 4 = \boxed{16\pi\sqrt{3}}$ (Note that there is another way to orient the cylinder that gives a larger volume; however, this second orientation does not fit inside the ellipsoid).

26. $\boxed{\textbf{B}}$: First, use the identity $\cos^2 x = 1 - \sin^2 x$ and substitute $x = \sin\theta$. This gives

$$I = \int_0^{\frac{\pi}{2}}\cos^3\theta\ln(\sin\theta)d\theta = \int_0^{\frac{\pi}{2}}\cos\theta(1-\sin^2\theta)\ln(\sin\theta) = \int_0^1 \ln x\,dx - \int_0^1 x^2\ln x\,dx$$

Then, performing integration by parts (using $u = \ln x$ and $v = dx$ for the first integral and $u = \ln x$ and $dv = x^2 dx$ for the second) gives

$$I = \Big((x\ln x - x) - \Big(\frac{x^3}{3}\ln x - \frac{x^3}{9}\Big)\Big)\Big|_0^1$$

At $x = 1$, this evaluates to $-\frac{8}{9}$; since $\ln x$ is undefined at $x = 0$, we can do the following:

$$\lim_{x\to 0} x^3 \ln x = \lim_{x\to 0} \frac{-\ln(1/x)}{1/x^3} = \lim_{u\to\infty} \frac{-\ln u}{u^3} = 0$$

The same applies to $\lim\limits_{x\to 0} x\ln x$, so at $x = 0$, the integral evaluates to equal 0, giving $I = \boxed{-\frac{8}{9}}$

27. $\boxed{\text{A}}$: $x^2 + y^2 = 4 \Rightarrow y = \pm\sqrt{4-x^2} \Rightarrow L = x + 3y = 3y + \sqrt{4-y^2}$. Then, when L is at a relative maximum, $L' = 3 + \frac{-2y}{2\sqrt{4-y^2}} = 0 \Rightarrow 3\sqrt{4-y^2} = y \Rightarrow 36 - 9y^2 = y^2 \Rightarrow y = \frac{6}{\sqrt{10}}$. From this, we get that $x = \frac{2}{\sqrt{10}}$ and $L = x + 3y = \frac{20}{\sqrt{10}} = \boxed{2\sqrt{10}}$

28. $\boxed{\text{B}}$: Using integration by parts, we get that $\int_0^\infty t^3 e^{-t} dt = \boxed{6}$. Alternatively, since by definition, $\Gamma(z) = (z-1)!$, $\Gamma(4) = 3! = 6$.

29. $\boxed{\text{A}}$: $\lim\limits_{x\to 1} \frac{\Gamma(x+1)-1}{\Gamma(x)-1} = \lim\limits_{x\to 1} \frac{\int_0^\infty t^x e^{-t} dt - 1}{\int_0^\infty t^{x-1} e^{-t} dt - 1}$. By definition, $\Gamma(z) = (z-1)!$, so $\Gamma(2) = \Gamma(1) = 1$, meaning that our limit is an indeterminate form. We can then use L'Hopital's rule and differentiation under the integral sign:

$$\lim_{x\to 1} \frac{\int_0^\infty t^x e^{-t} dt - 1}{\int_0^\infty t^{x-1} e^{-t} dt - 1} \Rightarrow \lim_{x\to 1} \frac{\int_0^\infty t^x e^{-t} \ln x \, dt}{\int_0^\infty t^{x-1} e^{-t} \ln x \, dt} = \frac{\int_0^\infty t e^{-t} \ln x \, dt}{\int_0^\infty e^{-t} \ln x \, dt}$$

.

The bottom integral is simply $-\gamma$; then, using integration by parts with $u = t\ln t$ and $t = e^{-t} dt$, we obtain:

$$\int_0^\infty t e^{-t} \ln x \, dt = -t \ln t e^{-t}\Big|_0^\infty + \int_0^\infty e^{-t} \ln t dt + \int_0^\infty e^{-t} dt = 1 - \gamma$$

Thus, our limit is equal to $\frac{1-\gamma}{-\gamma} = \boxed{1 - \frac{1}{\gamma}}$.

30. $\boxed{\text{C}}$: $u = \frac{1}{\sin x}$. Then, as x approaches 0, u will approach ∞ and $-\infty$ from the right and left sides, respectively. Thus, this limit can be rewritten as $\lim\limits_{x\to\infty} (1 + \frac{8}{x})^{2x}$, which equals $\boxed{e^{16}}$.

Alternatively, $\lim\limits_{x\to 0} (1 + 8\sin x)^{2\csc x} = e^{\lim_{x\to 0} \frac{2\ln(1+8\sin x)}{\sin x}} \Rightarrow e^{\lim_{x\to 0} \frac{\frac{16\cos x}{1+8\sin x}}{2\cos x}} = e^{16}$ (through L'Hopital's Rule).