

**Answer Key:**

**1. A**

**2. A**

**3. C**

**4. A**

**5. A**

**6. C**

**7. C**

**8. E**

**9. C**

**10. D**

**11. C**

**12. A**

**13. D**

**14. B**

**15. A**

**16. B**

**17. B**

**18. D**

**19. A**

**20. A**

**21. A**

**22. B**

**23. A**

**24. B**

**25. C**

**26. B**

**27. C**

**28. A**

**29. A**

**30. A**

**Solutions:**

- 1. A:** L is the 4<sup>th</sup> letter alphabetically among the letters present, so there are  $3 \cdot 5! = 360$  permutations before all of the permutations that start with L. All of the LEG\_\_ permutations are first among the L\_\_ permutations, so it suffices to arrange the OSI, which we can simply write manually: IOS, ISO, OIS, OSI, so LEGOSI appears as the  $\boxed{364}$ <sup>th</sup> entry.
- 2. A:** Letting the common difference of the sequence be  $d$ , we have  $-20 = a_{25} = a_{20} + 5d = 25 + 5d$ . Solving gives  $d = -9$ , so  $a_1 = a_{20} - 19d = 25 - 19(-9) = \boxed{196}$ .
- 3. C:** Letting the common ratio of the sequence be  $r$ , we have  $4 = g_5 = g_3 \cdot r^2 = 9r^2$ . Solving gives  $r = \pm \frac{2}{3}$ , and since  $g_{10} < 0$ , we must have  $r = -\frac{2}{3}$ . We can then compute  $g_1 = \frac{g_3}{r^2} = \frac{81}{4}$ , so the formula for the sum of an infinite geometric series gives  $\frac{81/4}{1 - (-2/3)} = \boxed{\frac{243}{20}}$ .
- 4. A:** The ratio test gives  $\lim_{n \rightarrow \infty} \frac{1/n}{1/(n+1)} = \lim_{n \rightarrow \infty} \frac{n+1}{n} = 1$ , so the ratio test is inconclusive. The root test gives  $\lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{n}} = 1$  (this can be shown using the standard log and L'Hopital argument), so the root test is inconclusive. The  $n$ th term test has  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ , so the  $n$ th term test is inconclusive. The integral test gives  $\int_1^{\infty} \frac{1}{x} dx = [\ln(x)]_1^{\infty}$ , which diverges, so the integral test shows that the series diverges.
- 5. A:** This is sequence [A000213](#) in the OEIS; the Tribonacci number starting with 1,1,1. The pattern is that after the first three terms, each term is the sum of the three previous terms. The next term is thus  $57 + 105 + 193 = \boxed{355}$ .
- 6. C:** This is sequence [A057944](#) in the OEIS; where, indexing starting at 0, the  $n$ th term is the largest triangular number less than or equal to  $n$  (or equivalently, starting at 1, the  $n$ th term is the largest triangular number strictly less than  $n$ ). This is asking for the entry in the series with index 15, which is  $\boxed{15}$ , as 15 is a triangular number.
- 7. C:** This is sequence [A006715](#) in the OEIS; a look-and-say sequence with first term 3. The pattern here is that each term after the first describes the number before it in the sequence. For instance, 3 contains "one 3", so the next term is 13. 13 has "one 1, one 3", so the next term is 1113. This has "three 1s, one 3", so we then have 3113. This is "one 3, two 1s, one 3", so we have 132113. We are interested in the term following 1113122113, which is "three 1s, one 3, one 1, two 2s, two 1s, one 3", or  $\boxed{311311222113}$ .
- 8. E:** This is sequence [A010861](#) in the OEIS; the constant sequence with value 22, so the next term is  $\boxed{22}$ . One notable property about this constant sequence is that it is also a look-and-say sequence, and indeed the *only* look-and-say sequence that does not grow to infinity!
- 9. C:** Recalling that  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$  for  $-1 < x < 1$ , the order 4 Maclaurin polynomial is  $1 + x + x^2 + x^3 + x^4$  (one could also compute the derivatives and use the definition to get this). Plugging in  $\frac{1}{2}$  gives  $\boxed{\frac{31}{16}}$ .

**10. D:** The given factorization suggests partial fraction decomposition, so we write  $\frac{n}{n^4+n^2+1} = \frac{An+B}{n^2+n+1} + \frac{Cn+D}{n^2-n+1}$ . Clearing denominators gives  $(An+B)(n^2-n+1) + (Cn+D)(n^2+n+1) = n$ . Expanding and matching coefficients gives  $A+C=0$ ,  $-A+B+C+D=0$ ,  $A-B+C+D=1$ , and  $B+D=0$ . Plugging the first and fourth equations into the third and second respectively simplifies to  $-A+C=0$  and  $-B+D=1$ , so we can get  $A=C=0$ ,  $B=-\frac{1}{2}$ , and  $D=\frac{1}{2}$ . Thus  $\frac{n}{n^4+n^2+1} = \frac{1/2}{n^2-n+1} - \frac{1/2}{n^2+n+1}$ . Hoping for telescoping, we can note that  $n^2-n+1 = (n-1)^2 + (n-1) + 1$ , so

$$\sum_{n=5}^N \left( \frac{1/2}{(n-1)^2 + (n-1) + 1} - \frac{1/2}{n^2 + n + 1} \right) = \frac{1/2}{4^2 + 4 + 1} - \frac{1/2}{N^2 + N + 1}.$$

As  $N \rightarrow \infty$ , this latter term approaches 0, so the desired sum is  $\boxed{\frac{1}{42}}$ .

**11. C:** Note that  $n^2 + n = n(n+1)$  is always even, so  $n^2 + n + 2025$  is always odd. Thus the desired sequence is simply  $-1 - 1 - 1 - 1 - \dots$ , which clearly  $\boxed{\text{diverges to infinity}}$ .

**12. A:** Since exponentials grow much faster than polynomials, by the limit comparison test it suffices to check the convergence of  $\sum_{n=1}^{\infty} \frac{(-2)^{3n}}{(-3)^{2n}} = \sum_{n=1}^{\infty} \left(\frac{-8}{9}\right)^n$ . This is a geometric series with ratio  $-1 < r < 1$ , so it  $\boxed{\text{converges absolutely}}$ .

**13. D:** Dividing numerator and denominator by  $2^n$ , the  $n$ th term of this sequence is  $\frac{(-1)^n}{1+n/2^n}$ , which in the limit alternates between 1 and  $-1$ , and in any case does not approach 0, so by the  $n$ th term test the series does not converge. We claim the series in fact  $\boxed{\text{bifurcates}}$ . Informally, this is because the tail behavior of the series is essentially just adding and subtracting 1, so we get the same behavior as  $\sum_{n=1}^{\infty} (-1)^n$  offset by some finite initial segment.

One potential worry is that positive and negative offsets are not quite by the same amount, and perhaps those errors add up enough to actually get to infinity. To dispel this concern, we can compute one of these "error" terms to be

$$\frac{1}{1 + \frac{n}{2^n}} - \frac{1}{1 + \frac{(n+1)}{2^{(n+1)}}} = \frac{\frac{1}{2^n} \left( \frac{(n+1)}{2} - n \right)}{\left( 1 + \frac{n}{2^n} \right) \left( 1 + \frac{(n+1)}{2^{(n+1)}} \right)} \sim \frac{n}{2^n}.$$

Summing these converges to some finite number, so indeed our partial sums are bounded.

**14. B:** Taking the absolute value, the integral test gives  $\int_2^{\infty} \frac{1}{x\sqrt{\ln(x)}} dx = \left[ 2\sqrt{\ln(x)} \right]_2^{\infty}$  (u-sub  $u = \ln(x)$ ), which diverges, so the series does not converge absolutely. However, note that  $x\sqrt{\ln(x)}$  is positive and decreases to 0, so by the alternating series test  $\boxed{\text{converges conditionally}}$ .

**15. A:** We can compute the first few terms to be  $0, 1, \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{11}{16}$ , and notice that the difference between adjacent terms cuts in half every time  $(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots)$  and has opposite signs  $(+1, -\frac{1}{2}, +\frac{1}{4}, -\frac{1}{8}, \dots)$ . We can verify both of these facts formally by induction, but less formally, the halving comes from the averaging in the recursive formula, while the alternating sign comes from one "endpoint" of the average getting replaced by the average, leading the other one to pull further in the next averaging. In any case, we thus get a geometric series with first term 1 and common ratio  $-\frac{1}{2}$ , for a sum of  $\frac{1}{1-1/2} = \boxed{\frac{2}{3}}$ .

**16. B:** Writing  $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ , we have  $x \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!}$ . By the definition of a Maclaurin series, the 2025<sup>th</sup> derivative of this is given by 2025! times the coefficient of the  $x^{2025}$  term of the series expansion, which is  $\frac{1}{2024!}$  in this case. Thus the 2025<sup>th</sup> derivative is  $\frac{2025!}{2024!} = \boxed{2025}$ .

**17. B:** Completing the square in the denominator gives  $(x - 3)^2 = -16 \rightarrow x = 3 \pm 4i$ . The distance from 0 to either of these is  $\boxed{5}$ .

**18. D:** First note that the solutions to  $\frac{x^6 - 1}{x - 1} = 0$  are the sixth roots of unity other than 1, so all  $P_n$ 's are pentagons. Further note that in the process of going from  $P_n$  to  $P_{n+1}$ , the sum of the five vertices of the polygon never changes (when viewed as complex numbers). We can compute this sum for  $P_1$  to be  $-1$ , as the sum of the sixth roots of unity is 0 (by Vieta's or symmetry about the unit circle, for instance) and then we subtract off 1. This corresponds to the point  $(-1, 0)$ , so we end up with five vertices that sum to this, so each one has to be  $\boxed{\left(-\frac{1}{5}, 0\right)}$ .

**19. A:** We want to compute  $f(x) = \sum_{n=0}^{\infty} F_n x^n = F_0 + F_1 x + F_2 x^2 + F_3 x^3 + F_4 x^4 + F_5 x^5 + \dots$ . Using the recurrence relation on each term starting with the  $x^2$  term gives  $F_0 + F_1 x + (F_0 + F_1)x^2 + (F_1 + F_2)x^3 + \dots$ . Taking the first term of each decomposed sum gives  $F_0 x^2 + F_1 x^3 + F_2 x^4 + \dots = x^2 f(x)$ , while adding an  $F_0 x$  (i.e. 0) to taking the second term gives  $F_0 x + F_1 x^2 + F_2 x^3 + \dots = x f(x)$ . We then have  $f(x) = x + x^2 f(x) + x f(x)$ , and solving for  $f(x)$  gives  $f(x) = \frac{x}{1 - x - x^2}$ . Plugging in  $x = \frac{1}{3}$  gives  $\boxed{\frac{3}{5}}$ .

**20. A:** Inspired by the hint (or perhaps by manually computing the first few digits to be 0.011235 ...), we see if we can connect the value of  $\frac{1}{89}$  to the  $f(x)$  in the previous problem. Note that  $f\left(\frac{1}{10}\right) = \frac{10}{89}$ , so  $\frac{1}{89} = \frac{1}{10} f\left(\frac{1}{10}\right) = \frac{F_0}{10} + \frac{F_1}{10^2} + \frac{F_2}{10^3} + \dots$ , confirming our earlier observed pattern of Fibonacci numbers in the digits. The 12<sup>th</sup> digit after the decimal point will be affected by terms starting at the  $\frac{F_{11}}{10^{12}}$  term (the  $\frac{1}{10^{13}}$  term and such will also affect it due to carries).

We compute the relevant Fibonacci numbers (starting at  $F_1$ , separating every 5 for readability) to be 1,1,2,3,5 | 8,13,21,34,55 | 89,144,233,377,610. We're interested in  $\frac{89}{10^{12}} + \frac{144}{10^{13}} + \frac{233}{10^{14}} + \frac{377}{10^{15}} = \frac{106.107}{10^{12}}$ , which suggests the 12<sup>th</sup> digit is  $\boxed{6}$ , as the remaining terms are too small to induce a further carry. (The next one adds an additional 0.1 to the numerator, and the rest decay exponentially with a factor  $< \frac{1}{5}$ ).

**21. A:** From problem 19 we have  $\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}$ . Noting that the  $F_0$  term is 0 and integrating gives  $\sum_{n=1}^{\infty} \frac{F_n x^{n+1}}{n+1} = \int_0^x \frac{t}{1 - t - t^2} dt$ . To get an  $n$  in the denominator instead, we can divide by  $x$  before integrating to get  $\sum_{n=1}^{\infty} \frac{F_n x^n}{n} = \int_0^x \frac{1}{1 - t - t^2} dt$ . To get our desired expression, we can combine these and get

$$\sum_{n=1}^{\infty} F_n x^n \left( \frac{1}{n} + \frac{1}{n+1} \right) = \int_0^x \frac{1}{1 - t - t^2} dt + \frac{1}{x} \int_0^x \frac{t}{1 - t - t^2} dt.$$

Plugging in  $x = \frac{1}{2}$  gives that our desired sum is equal to  $\int_0^{1/2} \frac{1+2t}{1-t-t^2} dt$ . Letting  $u = 1 - t - t^2$ ,  $du = (-1 - 2t) dt$  gives  $\int_{1/4}^{1/4} -\frac{1}{u} du = \int_{1/4}^1 \frac{1}{u} du = [\ln(u)]_{1/4}^1 = \boxed{2 \ln 2}$ .

22. B: Writing  $\binom{2n}{n} = \frac{2n!}{n!n!}$ , the ratio test gives

$$\lim_{n \rightarrow \infty} \frac{\frac{(2n+2)!}{(n+1)!^2} x^{n+1}}{\frac{(2n)!}{n!^2} x^n} = \lim_{n \rightarrow \infty} \frac{2(2n+1)}{n+1} x = 4x.$$

This has absolute value less than 1 for  $|x| < \frac{1}{4}$ , and absolute value greater than 1 for  $|x| > \frac{1}{4}$ , and absolute value equal to 1 for  $x = \pm \frac{1}{4}$ . The series thus converges in the first case, diverges in the second case, and is inconclusive in the third. For this case, we'll use the more precise Stirling's approximation and the limit comparison test. We have

$$\frac{(2n)!}{n!^2} \left(\frac{1}{4}\right)^n \sim \frac{\sqrt{4\pi n} \left(\frac{2n}{e}\right)^{2n}}{2\pi n \left(\frac{n}{e}\right)^{2n}} \left(\frac{1}{4}\right)^n = \frac{1}{\sqrt{\pi n}}$$

By the  $p$ -series test this does not converge absolutely, so  $\frac{1}{4}$  is not in the domain, but  $\frac{1}{\sqrt{\pi n}}$  is positive and decreases to 0, so by the alternating series test the series for  $-\frac{1}{4}$  does converge conditionally. Thus the domain of the generating function is  $\left[-\frac{1}{4}, \frac{1}{4}\right)$ .

23. A: Following the hint, we consider the expansion of  $(1+x)^{-1/2}$ . Using the generalized binomial theorem (or taking derivatives), we can compute that  $(1+x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} 1^{-1/2-n} x^n$ , where  $\binom{-1/2}{n} = \frac{(-\frac{1}{2})(-\frac{1}{2}-1)\dots(-\frac{1}{2}-(n-1))}{n!}$ . Simplifying this, we get  $\frac{(-1)(-3)\dots(-2n-1)}{2^{2n}n!} = \frac{(-1)^n(2n-1)!!}{2^{2n}n!}$ . If we multiply in all of the even terms in the numerator and denominator, we end up with  $\frac{(-1)^n(2n)!}{4^n n!^2} = \left(-\frac{1}{4}\right)^n \binom{2n}{n}$ , so our original series is  $\sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n \binom{2n}{n} x^n$ . This is almost what we want, but that  $\left(-\frac{1}{4}\right)^n$  term is in the way. To deal with that, we can replace  $x$  with  $-4x$ , yielding  $(1-4x)^{-1/2} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n$ . Noting that the LHS is an increasing function in  $x$ , we want to plug in as large of an  $x$  as possible (of the form  $\frac{1}{k}$  for integer  $k$ ). Using the domain from the previous problem, this largest  $x$  has to be  $\frac{1}{5}$ , yielding a sum of  $\sqrt{5}$ .

24. B: From the previous problem, we have  $(1-4x)^{-1/2} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n$ . Integrating both sides yields

$$\sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^{n+1} = \int_0^x (1-4t)^{-1/2} dt = \left[-\frac{1}{2}(1-4t)^{1/2}\right]_0^x = \frac{1-\sqrt{1-4x}}{2},$$

And dividing by  $x$  on both sides gives  $\sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n = \frac{1-\sqrt{1-4x}}{2x}$ . One can verify this is an increasing function in  $x$  (or just plug in possible values of  $k$ ; there aren't that many). However, instead of plugging in  $\frac{1}{5}$  as before, note that by introducing another factor of  $\frac{1}{n}$ , our limit comparison from problem 22 becomes  $\frac{1}{n\sqrt{n}}$  which does converge absolutely, so  $\frac{1}{4}$  is in the domain this time. Therefore, we can plug in  $\frac{1}{4}$  to get our answer of  $\boxed{2}$ .

**25. C:** The given order of summation is intractable, so we switch the order to get  $\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{n}{m^n}$  (which we can do since everything converges absolutely).

We first compute  $\sum_{n=2}^{\infty} \frac{n}{m^n}$ . For fixed  $n$ . Let the LHS be equal to  $S$ . We then have  $\frac{S}{m} = \sum_{n=2}^{\infty} \frac{n}{m^{n+1}} = \sum_{n=3}^{\infty} \frac{n-1}{m^n}$ . Subtracting this from the original gives  $S - \frac{S}{m} = \frac{2}{m^2} + \sum_{n=3}^{\infty} \frac{1}{m^n}$ , the latter of which is an infinite geometric series with sum  $\frac{(\frac{1}{m})^3}{1 - \frac{1}{m}} = \frac{1}{m^2(m-1)}$ . Therefore  $S = \left(\frac{m}{m-1}\right) \left(\frac{2}{m^2} + \frac{1}{m^2(m-1)}\right) = \frac{2m-1}{m(m-1)^2}$ .

Now we wish to compute  $\sum_{m=2}^{\infty} \frac{2m-1}{m(m-1)^2}$ . Partial fraction decomposition gives us the form  $\frac{A}{m} + \frac{B}{m-1} + \frac{C}{(m-1)^2}$  or  $A(m-1)^2 + Bm(m-1) + Cm = 2m-1$ . Expanding and matching coefficients gives  $A+B=0$ ,  $-2A-B+C=2$ , and  $A=-1$ , and we can solve to get  $B=1$ ,  $C=1$ , so  $\frac{2m-1}{m(m-1)^2} = -\frac{1}{m} + \frac{1}{m-1} + \frac{1}{(m-1)^2}$ . We can thus split our desired sum into  $\sum_{m=2}^{\infty} \left(\frac{1}{m-1} - \frac{1}{m}\right)$  and  $\sum_{m=2}^{\infty} \frac{1}{(m-1)^2}$ . The former telescopes to 1, while the latter is equal to  $\frac{\pi^2}{6}$  by the identity in the instructions. Thus our total sum is  $\boxed{\frac{\pi^2}{6} + 1}$ .

**26. B:** Seeing a  $\frac{1}{n}$ -like term in the denominator suggests integrating a power series, and indeed the given sum is equal to  $\sum_{n=0}^{\infty} \left[ \int_0^1 (-x)^{3n} dx \right]$ . Swapping the sum and integral (which we can do by, e.g., the dominated convergence theorem) gives the sum is equal to  $\int_0^1 (1-x^3+x^6-x^9+\dots) dx = \int_0^1 \frac{1}{1+x^3} dx$ .

To evaluate this, we factor  $1+x^3 = (1+x)(1-x+x^2)$  and use partial fraction decomposition. We want  $\frac{1}{(1+x)(1-x+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1-x+x^2}$ , or  $1 = A(1-x+x^2) + (Bx+C)(1+x)$ . Expanding and matching coefficients gives  $A+B=0$ ,  $-A+B+C=0$ , and  $A+C=1$ . Solving gives  $A = \frac{1}{3}$ ,  $B = -\frac{1}{3}$ , and  $C = \frac{2}{3}$ , so  $\frac{1}{1+x^3} = \frac{1}{3} \left( \frac{1}{1+x} + \frac{2-x}{1-x+x^2} \right)$ . We can write the second term as  $\frac{\frac{1}{2}-x}{1-x+x^2} + \frac{\frac{3}{2}}{1-x-x^2}$ , and now each term is ready to integrate.

$$\int_0^1 \frac{1}{1+x} dx = [\ln(1+x)]_0^1 = \ln(2).$$

$$\int_0^1 \frac{\frac{1}{2}-x}{1-x+x^2} dx = \left[ -\frac{1}{2} \ln(1-x+x^2) \right]_0^1 = 0.$$

$$\int_0^1 \frac{\frac{3}{2}}{1-x+x^2} dx = \frac{3}{2} \int_0^1 \frac{1}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}} dx = \left[ \frac{3}{2} \cdot \frac{2}{\sqrt{3}} \arctan\left(\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right) \right]_0^1 = \frac{\pi\sqrt{3}}{3}.$$

Adding these and dividing by 3 gives  $\frac{1}{9}\pi\sqrt{3} + \frac{1}{3}\ln(2)$ , so  $p = \frac{1}{9}$  and  $q = \frac{1}{3}$ , for a sum of  $\boxed{\frac{4}{9}}$ .

27. C: We can compute

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^4 = \begin{pmatrix} 1 & 4 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix},$$

so we might conjecture that

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix},$$

which can be proved by induction. As such, the sum of the entries as the given sum is  $\sum_{n=0}^{\infty} \frac{3 + \frac{5n+n^2}{2}}{n!}$ .

To compute this, it suffices to compute  $\sum_{n=0}^{\infty} \frac{1}{n!}$ ,  $\sum_{n=0}^{\infty} \frac{n}{n!}$ ,  $\sum_{n=0}^{\infty} \frac{n^2}{n!}$ . The former is just  $e$  by definition, and the second is equal to  $\sum_{n=1}^{\infty} \frac{1}{(n-1)!} = e$  as well. For the third, we have  $\sum_{n=1}^{\infty} \frac{n}{(n-1)!} = \sum_{n=1}^{\infty} \frac{n-1}{(n-1)!} + \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \sum_{n=2}^{\infty} \frac{1}{(n-2)!} + \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = 2e$ . Putting all this together gives the desired sum is  $\boxed{\frac{13e}{2}}$ .

28. A: We compute the first few terms to be 0,1,2,4,6,9,12,16,20,25, ..., and conjecture that  $a_{2n} = n(n+1)$  and  $a_{2n-1} = n^2$  for all  $n \geq 0$ , which can be proved by induction. Thus  $a_{4048} = 2024(2025)$ , so we're interested in finding two numbers of the form  $m^2$  or  $m(m+1)$  that multiply to 2024(2025). Noting that  $2025 = 45^2$  and thus  $2024 = 45^2 - 1 = 44 \cdot 46$ , we have  $2024(2025) = (44(45))(45(46))$ , which is equal to  $a_{88} \cdot a_{90}$ , summing to 178. Note that we can also factor out 2s from the 44 and 46 to get  $2024(2025) = (22(23))(90^2)$ , which is equal to  $a_{44} \cdot a_{179}$ , summing to 223. This gives an answer of  $178 + 223 = \boxed{401}$ .

29. A: Rewrite as  $\int_0^{\pi/2} \frac{\ln(\cos(x)) \cos(x)}{\sin(x)} dx = \int_0^{\pi/2} \frac{\ln(\cos(x)) \cos(x) \sin(x)}{\sin^2(x)} dx = \int_0^{\pi/2} \frac{\ln(\cos(x)) \cos(x) \sin(x)}{1-\cos^2(x)} dx$ .

Substituting  $u = \cos(x)$ ,  $du = -\sin(x) dx$  gives  $\int_1^0 \frac{u \ln(u)}{1-u^2} (-du) = \int_0^1 \frac{u \ln(u)}{1-u^2} du$ . By the series expansion of  $\frac{1}{1-u^2}$ , this is equal to  $\int_0^1 \sum_{n=0}^{\infty} u^{2n+1} \ln(u) du$ . Swapping the sum and integral (dominated convergence), this is equal to  $\sum_{n=0}^{\infty} \int_0^1 u^{2n+1} \ln(u) du$ . We can integrate by parts to get  $\int_0^1 u^{2n+1} \ln(u) du = \left[ \frac{1}{2n+2} u^{2n+2} \ln(u) \right]_0^1 - \int_0^1 \frac{1}{(2n+2)} u^{2n+2} \left( \frac{1}{u} \right) du = -\frac{1}{2(n+1)} \int_0^1 u^{2n+1} du = -\frac{1}{4(n+1)^2}$ . Therefore the desired sum is equal to  $-\frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \boxed{-\frac{\pi^2}{24}}$  from the identity in the instructions.

30. A: We want to integrate by parts, so we first calculate the derivative of  $\text{Li}_2(x)$ . Differentiating the series expansion gives  $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n} = \frac{1}{x} \sum_{n=1}^{\infty} \frac{x^n}{n} = \frac{-\ln(1-x)}{x}$ . It then follows that  $\int_0^1 \text{Li}_2(x) dx = [x \text{Li}_2(x)]_0^1 - \int_0^1 x \left( \frac{-\ln(1-x)}{x} \right) dx = (\text{Li}_2(1) - 0) + \int_0^1 \ln(1-x) dx$ .  $\text{Li}_2(1) = \frac{\pi^2}{6}$  by the given identity, and letting  $u = 1-x$  and integrating by parts gives  $\int_0^1 \ln(1-x) dx = \int_0^1 \ln(x) dx = [x \ln(x)]_0^1 - \int_0^1 x \left( \frac{1}{x} \right) dx = -1$ , so the original integral is equal to  $\boxed{\frac{\pi^2}{6} - 1}$ .