

Important Instructions for this Test: The answer choice E: NOTA indicates that “None Of These Answers” are correct. At some point, it may be useful to know that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Good luck, and have fun!

1. Legosi is really bored one day and decides to write out all 720 permutations of “LEGOSI” in alphabetical order, where e.g. “EGILOS” is the first entry, “EGILSO” is the second, all the way up to “SOLIGE” as the 720th. In what position does “LEGOSI” appear in this sequence?

- A: 364 B: 361 C: 362 D: 363 E: NOTA

2. Suppose $\{a_n\}_n$ is an arithmetic sequence. If $a_{20} = 25$ and $a_{25} = -20$, compute a_1 .

- A: 196 B: 44 C: -146 D: 151 E: NOTA

3. Suppose $\{g_n\}_n$ is a geometric sequence. If $g_3 = 9$, $g_5 = 4$, and $g_{10} < 0$, compute $\sum_{n=1}^{\infty} g_n$.

- A: 27 B: $\frac{243}{4}$ C: $\frac{243}{20}$ D: $\frac{27}{5}$ E: NOTA

4. How many of the following tests can be used to prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges?

- (I) Ratio test (II) Root test
(III) n th term test (IV) Integral test

- A: 1 B: 2 C: 3 D: 4 E: NOTA

Please Use the Following Information to Answer Questions 5 to 8:

The On-Line Encyclopedia of Integer Sequences (OEIS) is a large database of sequences of integers that have shown up in some mathematical context. Each problem will feature an **initial segment** of a *unique* entry of the OEIS (that is, the first however-many entries of a sequence deemed well-defined enough to appear in the OEIS). Your goal is to find the next term in the sequence. Disputes of the form “oh well you can contrive a sequence such that X” will not be accepted; there are enough terms present to uniquely identify the sequence in the OEIS.

5. Find the next term in the sequence: 1, 1, 1, 3, 5, 9, 17, 31, 57, 105, 193, ____.

- A: 355 B: 298 C: 323 D: 279 E: NOTA

6. Find the next term in the sequence: 0, 1, 1, 3, 3, 3, 6, 6, 6, 6, 10, 10, 10, 10, ____.

- A: 10 B: 16 C: 15 D: 12 E: NOTA

7. Find the next term in the sequence: 3, 13, 1113, 3113, 132113, 1113122113, ____.
(Hint: each term after the first depends only on the previous term.)

- A: 31221123112 C: 311311222113 E: NOTA
B: 1213112112 D: 132112312113

8. Find the next term in the sequence: 22, 22, 22, 22, 22, 22, 22, 22, 22, 22, ____.

- A: 33 B: 44 C: 11 D: 23 E: NOTA

9. Collot is, for some reason, interested in approximating the value of 2. To do this, he takes a fourth-degree Maclaurin polynomial of $\frac{1}{1-x}$ and evaluates it at $x = \frac{1}{2}$. What value does he obtain for his approximation?

A: $\frac{15}{8}$

B: 2

C: $\frac{31}{16}$

D: $\frac{63}{32}$

E: NOTA

10. Given that $n^4 + n^2 + 1 = (n^2 + n + 1)(n^2 - n + 1)$, evaluate

$$\sum_{n=5}^{\infty} \frac{n}{n^4 + n^2 + 1}.$$

A: $\frac{1}{62}$

B: $\frac{1}{21}$

C: $\frac{1}{31}$

D: $\frac{1}{42}$

E: NOTA

Please Use the Following Information to Answer Questions 11 to 14:

Determine the convergence of each of the following series. For the purposes of these problems, a series “diverges to infinity” if the sequence of partial sums is unbounded (that is, its **absolute values** tend to ∞), while a series “bifurcates” if it does not converge, but the sequence of partial sums can be bounded on both sides by finite values.

11. Determine the convergence of

$$\sum_{n=1}^{\infty} (-1)^{n^2+n+2025}.$$

A: Converges absolutely

B: Converges conditionally

C: Diverges to infinity

D: Bifurcates

E: NOTA

12. Determine the convergence of

$$\sum_{n=1}^{\infty} \frac{(-2)^{3n} + n^3}{(-3)^{2n} + n^2}.$$

A: Converges absolutely

B: Converges conditionally

C: Diverges to infinity

D: Bifurcates

E: NOTA

13. Determine the convergence of

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{2^n + n}.$$

A: Converges absolutely

B: Converges conditionally

C: Diverges to infinity

D: Bifurcates

E: NOTA

14. Determine the convergence of

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt{\ln(n)}}$$

A: Converges absolutely

B: Converges conditionally

C: Diverges to infinity

D: Bifurcates

E: NOTA

15. A sequence $\{a_n\}_{n=1}^{\infty}$ is defined such that $a_1 = 0$, $a_2 = 1$, and for all $n \geq 3$, $a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$. Compute $\lim_{n \rightarrow \infty} a_n$.

- A: $\frac{2}{3}$ B: $\frac{1}{2}$ C: $\frac{5}{6}$ D: 1 E: NOTA

16. If $g(x)$ is the 2025th derivative of $f(x) = x \cos(x)$ with respect to x , compute $g(0)$.

- A: 2024 B: 2025 C: 2024! D: 2025! E: NOTA

17. A theorem of complex analysis states that the radius of convergence of the series expansion of an analytic function at $z = a$ (for a a complex number) is the distance in the complex plane between a and the nearest singularity (place where the function is undefined due to dividing by 0). Use this result to find the radius of convergence of the Maclaurin series of $\frac{1}{x^2 - 6x + 25}$.

- A: 3 B: 5 C: 6 D: 4 E: NOTA

18. A sequence of polygons $\{P_n\}_{n=1}^{\infty}$ in the complex plane is defined such that P_1 is the convex polygon with vertices at the complex solutions to $\frac{x^6 - 1}{x - 1} = 0$, and for all $n \geq 1$, P_{n+1} is the polygon with vertices at the midpoints of the sides of P_n . As $n \rightarrow \infty$, P_n “approaches” a single point (in that all vertices eventually converge into this point). Find this point. Express your answer using Cartesian coordinates.

- A: (0,0) B: (-1,0) C: $(-\frac{1}{6}, 0)$ D: $(-\frac{1}{5}, 0)$ E: NOTA

Please Use the Following Information to Answer Questions 19 to 24:

Given a sequence $\{a_n\}_{n=0}^{\infty}$, its *generating function* is defined pointwise by $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for all x such that the series converges (and undefined elsewhere). For example, the generating function of the sequence $\{1\}_{n=0}^{\infty}$ is $\frac{1}{1-x}$ with domain $(-1, 1)$.

19. The *Fibonacci sequence* $\{F_n\}_{n=0}^{\infty}$ is defined such that $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. If $f(x)$ is the generating function of the Fibonacci sequence, compute $f(1/3)$. (Hint: solve for $f(x)$ in general, using the recurrence relation to spot copies of $f(x)$ on the right-hand side.)

- A: $\frac{3}{5}$ B: $\frac{1}{5}$ C: $\frac{4}{9}$ D: $\frac{2}{9}$ E: NOTA

20. If $1/89$ is written out as a decimal, what is the 12th digit after the decimal point? (Hint: this follows the previous problem for a reason.)

- A: 6 B: 5 C: 7 D: 8 E: NOTA

21. Compute

$$\sum_{n=1}^{\infty} \frac{F_n}{2^n} \left(\frac{1}{n} + \frac{1}{n+1} \right).$$

- A: $2 \ln(2)$ B: $\ln(3)$ C: $\ln\left(\frac{9}{2}\right)$ D: $\ln\left(\frac{9}{4}\right)$ E: NOTA

22. Compute the domain of the generating function of $\left\{\binom{2n}{n}\right\}_{n=0}^{\infty}$. You may find Stirling's approximation $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$ to be useful.

- A: $\left(-\frac{1}{4}, \frac{1}{4}\right)$ B: $\left[-\frac{1}{4}, \frac{1}{4}\right)$ C: $\left[-\frac{1}{4}, \frac{1}{4}\right]$ D: $\left(-\frac{1}{4}, \frac{1}{4}\right]$ E: NOTA

23. If k is a nonzero integer, compute the largest possible (finite) value of

$$\sum_{n=0}^{\infty} \binom{2n}{n} \left(\frac{1}{k}\right)^n.$$

(Hint: What is the series expansion of $(1+x)^{-1/2}$?)

- A: $\sqrt{5}$ B: 4 C: 2 D: $\frac{1+\sqrt{5}}{2}$ E: NOTA

24. The *Catalan numbers* defined by $C_n = \frac{1}{n+1} \binom{2n}{n}$ are of great importance in combinatorics (I encourage you to look them up on your own time!), but for the purposes of this test their definition is sufficient. If k is a nonzero integer, compute the largest possible (finite) value of

$$\sum_{n=0}^{\infty} C_n \left(\frac{1}{k}\right)^n.$$

- A: $\frac{5-\sqrt{5}}{10}$ B: 2 C: $\sqrt{5}$ D: $\frac{5-\sqrt{5}}{2}$ E: NOTA

25. Evaluate

$$\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{n}{m^n}.$$

- A: $\frac{\pi^2}{3}$ B: $\frac{\pi^2}{6}$ C: $\frac{\pi^2}{6} + 1$ D: $\frac{\pi^2}{3} - 1$ E: NOTA

26. If

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1}$$

can be written as $p\pi\sqrt{3} + q\ln(2)$ for rational p and q , compute $p + q$.

- A: $\frac{7}{18}$ B: $\frac{4}{9}$ C: $\frac{1}{6}$ D: $\frac{5}{18}$ E: NOTA

27. Compute the sum of the entries of

$$\sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n$$

(this sum is known as a *matrix exponential*).

- A: $6e$ B: $7e$ C: $\frac{13e}{2}$ D: $\frac{15e}{2}$ E: NOTA

28. Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence satisfying $a_0 = 0$; $a_1 = 1$; a_{2n}, a_{2n+1} , and a_{2n+2} form an arithmetic sequence for all non-negative integers n ; and a_{2n-1}, a_{2n} , and a_{2n+1} form a geometric sequence for all positive integers n . There two ordered pairs (m, n) with $m > n > 1$ such that $a_m \cdot a_n = a_{4048}$. Compute the sum of all possible values of $m + n$.

- A: 401 B: 397 C: 405 D: 403 E: NOTA

29. Sequences and series can be used in evaluating integrals! Evaluate

$$\int_0^{\pi/2} \ln|\cos(x)| \cot(x) \, dx.$$

- A: $-\frac{\pi^2}{24}$ B: $-\frac{\pi^2}{12}$ C: $-\frac{\pi^2}{18}$ D: $-\frac{\pi^2}{6}$ E: NOTA

30. For $0 \leq x \leq 1$, the *dilogarithm* of x , denoted $\text{Li}_2(x)$, is defined via the series $\text{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$. Evaluate $\int_0^1 \text{Li}_2(x) \, dx$.

- A: $\frac{\pi^2}{6} - 1$ B: $\frac{\pi^2}{6}$ C: $\frac{\pi^2}{12}$ D: $\frac{\pi^2}{12} - \frac{1}{2}$ E: NOTA