

Answer Key:

1. A
2. B
3. D
4. C
5. A
6. C
7. C
8. B
9. D
10. D
11. A
12. E (3)
13. C
14. D
15. D
16. A
17. A
18. B
19. C
20. B
21. A
22. B
23. D
24. B
25. B
26. C
27. C
28. D
29. E (745)
30. B

Solutions:

1. **A:** $2025 \equiv 1 \pmod{22}$, so this is equivalent to $1^2 \pmod{22} \equiv 1$
2. **B:** If n has any prime factors other than 3 and 5, the product would have a minimum of 30 positive integral factors. As such, we can represent n as $3^a \times 5^b$, and $2025n$ as $3^{4+a} \times 5^{2+b}$, giving us a number with $(5+a)(3+b)$ positive integral factors. The only values that give us 28 factors are $a = 2$ and $b = 1$, giving us $n = 45$
3. **D:** We can find $2025!$ has $(675 + 225 + 75 + 25 + 8 + 2) = 1010$ 3's, and $(405 + 81 + 16 + 3) = 505$ 5's. From this we get $a = b = 505$, and $a + b = 1010$
4. **C:** $12_3 = 5$, $23_4 = 11$, and $34_5 = 19$, so this is equal to $5 \times 11 + 19 = 74$
5. **A:** $600 = 2^3 \times 3^1 \times 5^2$. Since we are only counting factors not divisible by squares, this can be reduced to the factors of $2^1 \times 3^1 \times 5^1$, of which there are 8.
6. **A:** We are looking for a positive integer that satisfies $x \equiv 1 \pmod{2}$, $x \equiv 1 \pmod{3}$, $x \equiv 1 \pmod{5}$, and $x \equiv 2 \pmod{7}$. The first three can be simplified into $x \equiv 1 \pmod{30}$, and from there $x \equiv 121 \pmod{210}$. As we know the total number is between 800 and 1000, so she has 961 cupcakes. She can use 31 Bix Boxes with 31 cupcakes each.
7. **C:** $2025 = 3^4 \times 5^2$, so the sum of factors is $(1 + 3 + 9 + 27 + 81)(1 + 5 + 25) = 3751$
8. **B:** $S(7) = 8$, $S(8) = 15$, $S(15) = 24$, $S(24) = 60$
9. **D:** $S(p)$ for any prime number p is $p + 1$, so we can simply look for a composite number n where $S(n)$ is 1 more than a prime number. The first number that satisfies this is 6, with $S(6) = S(11)$, giving us $|6 - 11| = 5$
10. **D:** This is trivially true for $x = 1$, and since the sum of the factors of a perfect number is twice itself, this must also be true for perfect numbers. Quickly checking shows no values less than 28 work, so the sum of the 3 smallest is $1 + 6 + 28 = 35$

11. A: The units digit of 3^n loops in a pattern of 3,9,7,1, and as 14^{15} is a multiple of 4, the units digit must be 1

12. E: We can find the last 3 digits of a base 10 number by taking it modulo 1000, so we are solving $7^{7^{7^7}} \bmod 1000$. $7^{20} \bmod 1000 \equiv 1$, so we can solve this as $7^x \bmod 1000$ where $x \equiv 7^{7^7} \bmod 20$. $7^4 \bmod 20 \equiv 1$, so this can be solved by $7^y \bmod 20$, where $y \equiv 7^7 \bmod 4$, which gives us $y = 3$, $x = 3$, so the last 3 digits are $7^3 \bmod 1000 \equiv 343$, for a hundreds digit of 3

13. C: In order for $\frac{1}{n}$ to be a terminating decimal in prime base p , n must be able to be written as p^x , for a non-negative integer x . This gives us 6 values for $p = 2$, 4 for $n = 3$, 2 for $n = 5, 7$, and 1 each for all 21 primes in the range (10,100). Including 1 more solution for $n = 1$, we get $1 + 6 + 4 + 2 + 2 + 21 = 36$ solutions

14. D: Using long division we can get $A = 153846$, and factoring we get $A = 2 \times 3^3 \times 7 \times 11 \times 37$ for 5 distinct prime factors

15. D: Since $51^2 - 101 = 50^2$, we can rewrite this as $16^4 + 4(5^4)$. Using Sophie Germain factorization, we can see this is equal to $(16^2 + 2(16)(5) + 2(5)^2)(16^2 - 2(16)(5) + 2(5)^2) = 466 \times 146 = 2^2 \times 73 \times 233$, with sum of distinct prime factors being $2 + 73 + 233 = 308$

16. A: Solving for the minimum values to get $\{0,1,2,3,4\} \bmod 5$ using only 9 and 11, we get $\{0,11,22,18,9\}$. Adding $5n$ to each of these covers all numbers congruent modulo 5, so the largest impossible score would be 5 less than the largest of these, 22, which is 17

17. A: Writing out the numbers as polynomials we get: $2x^2 + 2x + 5 = (x + 1)^2 + 4(x + 1) + 2 + 6(x - 2) + 1$, which can be simplified to $x^2 - 10x + 9 = 0$, giving us solutions of 1 and 9. As 1 cannot be the base, we get that $x = 9$

18. B: For each power of 5 up to 5^4 , the coefficient can be 1, 0, or -1 . This is akin to the base-5 representation of the integer containing only 1, 0, or 4. Since $2025 = 31100_5$, there are 2 possibilities for the leading digit and 3 possibilities for every successive digit. Multiplying these together gives 162 possibilities. However, the 00000 case should be ignored, for 161 positive integers.

19. C: Solving for a , we can get $a = \frac{2b}{b-2}$, which can be rewritten as $a = 2 + \frac{4}{b-2}$. For a to be an integer, $\frac{4}{b-2}$ must be an integer, which means $(b-2)$ must evenly divide 4. 4 can be evenly divided by 6 integers, being $\pm 1, \pm 2, \pm 4$, with one of these being an extraneous solution of $a = b = 0$, for a total of 5 solutions.

20. B: For any given integer x , if $x \equiv 0 \pmod{5}$, $x^2 \equiv 0 \pmod{5}$, if $x \equiv 1, 4 \pmod{5}$, $x^2 \equiv 1 \pmod{5}$, and if $x \equiv 2, 3 \pmod{5}$, $x^2 \equiv 4 \pmod{5}$. Since every value modulo 5 is equally like on the interval $[1, 100]$, we can state that $x^2 \pmod{5}$ has a $\frac{1}{5}$ chance to be 0, and a $\frac{2}{5}$ chance each to be 1 or 4. The sum of 2 squares is $0 \pmod{5}$ if their remainders are $(0,0)$, $(1,4)$, or $(4,1)$, with the sum of probabilities being $\left(\frac{1}{5}\right)\left(\frac{1}{5}\right) + 2\left(\frac{2}{5}\right)\left(\frac{2}{5}\right) = \frac{9}{25}$

21. A: We can see that since $5y$, $10z$, and 200 are all multiples of 5, x must also be a multiple of 5. From here, we can set up an ordered list of all possible values of x , counting the possible values that (y, z) can be:

x	# of ordered pairs (y, z)
90	1
85	2
80	3
...	...
5	18

We can see the total number of solutions is $1 + 2 + 3 + \dots + 18 = 171$

22. B: Finding the units digit in base p is equivalent to taking the number modulo p , so we can write this as $(p+2)^{p+2} \pmod{p}$, which can be simplified to $2^{p+2} \pmod{p}$. Fermat's little theorem gives us $2^{p-1} \pmod{p} \equiv 1$, so we can rewrite this as $(2^{p-1} \times 2^3) \pmod{p} \equiv 8 \pmod{p}$. This gives us a units digit of 2 when $p = 3$, 3 when $p = 5$, 1 when $p = 7$, and 8 for any prime greater than 7, for a sum of all possible values of $2 + 3 + 1 + 8 = 14$

23. D: The greatest common factor must be a factor of the least common multiple, meaning it must also be a factor of 18, giving us 6 gcd, lcm pairs of $(1,19), (2,20), (3,21), (6,24), (9,27), (18,36)$. Each of these pairs can work as a pair of numbers with the listed gcd and lcm, as well as one more satisfying $(2,20)$ being 4 and 10, for a total of 7 pairs of numbers.

24. B: This is easiest solved with the Euclidean algorithm, taking the smallest 2 numbers in a set and adding their positive difference to the set. From this, we can get $(69733, 31921)$,

(37812,31921), (31921,5891), (5891,2466), (2466,959), (959,548), (548,411), (411,137)
 As 137 is prime, we can confirm it is a factor by dividing one of the original numbers, so our answer is $1 + 3 + 7 = 11$

25. B: We are looking for sequential numbers of up to 3 digits. We can group them based off of the number of digits in the terms of the sequence, with 3-digit numbers being in the form (1,1,1) or (0,1,2), and 2-digit numbers being in the form (0,1,1), with a 0-digit number referring to a sequence beginning with a 0. We can count each of these separately, with (0,1,1) and (0,1,2) representing 4 and 5 sequences respectively. A sequence in the form (1,1,1) can begin with any digit 1-9, and end with either any even number or any odd number, depending on the starting digit, for 5 options, giving us $9 \times 5 = 45$ sequences in the form (1,1,1), and $4 + 5 + 45 = 54$ total sequential numbers

26. C: n being on the interval $[1,100]$ limits us to a maximum of a 4-digit number, with the exception of $n = 100$ which is not a solution. This adds sequences in the form (0,2,2), (1,1,2), (2,1,1), and (1,1,1,1). We can quickly check that no sequence in the form (1,1,1,1) forms a square number, meaning all solutions must be generated from a 3 term sequence. Given this, we can know n^2 must be a multiple of 3, and therefore 9, meaning the middle term of the sequence must also be a multiple of 3. (0,2,2) cannot generate any square numbers, as every number it generates must be a multiple of 102, so we know the middle term of the sequence must be 1 digit. Given this, we can check sequences with a middle term of 3, 6, or 9 to find the 3 solutions of $n = \{6, 36, 54\}$ for the sequential numbers 36 (0,3,6), 1296 (12,9,6), and 2916 (2,9,16) for an answer of $6 + 36 + 54 = 96$

27. C: The fraction can be simplified to $x = n^2 + 6n + 3 + \frac{7}{n+2}$. For x to be a prime number, it first must be an integer, meaning $\frac{7}{n+2}$ must be an integer, giving us $n = \{-9, -3, -1, 5\}$. Plugging in the values gives us $\{29, -13, 5, 59\}$, and adding the prime values gives $29 + 5 + 59 = 93$

28. D: We can rewrite the equation to get $10^B = C^2 - A^2$, giving us a difference of squares, so $10^B = (C + A)(C - A)$. Because A, B, and C must be positive integers, we can know $(C + A)$ and $(C - A)$ must be factors of 10^B , and that A specifically is half the difference between these factors. Because A is an integer, this means that the difference between the 2 factors is an even number, meaning they are both even or both odd, and because B is a positive integer, we can know that they are both even, as their product must then be even. As A will be growing much faster than B, we will want to minimize A to minimize $|A - B|$, meaning we want to find the closest 2 even factors of 10^B . We can split this into 2 cases, with B being either even or odd, with exceptions for $B = 1$, where there are no solutions, and $B = 2$, with the only solution for (A, B, C) being (24, 2, 26).

For even values of B , we can represent it as $B = 2n$, for $n \geq 2$, with the 2 closest even factors of 10^B being $5^n 2^{n+1}$ and $5^n 2^{n-1}$, giving $A = \frac{5^n 2^{n+1} - 5^n 2^{n-1}}{2} = 5^n (2^n - 2^{n-2}) = \frac{3}{4} 10^n$, and

$$|A - B| = \frac{3}{4} 10^n - 2n$$

For odd values of B , we can represent it as $B = 2n + 1$, for $n \geq 1$, with the 2 closest even factors of 10^B being $5^{n+1} 2^n$ and $5^n 2^{n+1}$, giving $A = \frac{5^{n+1} 2^n - 5^n 2^{n+1}}{2} = 5^n (5 \times 2^{n-1} - 2^n) = \frac{3}{2} 10^n$, and

$$|A - B| = \frac{3}{2} 10^n - 2n - 1$$

We can see that as B increases, $|A - B|$ increases, so the minimum value must occur at $B = \{2, 3, 4\}$, and we can find the minimum is at $(15, 3, 35)$ with $|A - B| = 12$

29. E: Any positive integer modulo 9 is congruent to the sum of its digits modulo 9 when written in base 10. As such, we can find $88^{88} \bmod 9 \equiv 7$, which matches none of the answer choices, so the answer must be E.

30. B: 2025 has 15 positive divisors with a geometric mean of 45. The product of all divisors is therefore $(45)^{15}$.