

Answer Key:

1. A
2. B
3. B
4. D
5. D

6. B
7. C
8. A
9. D
10. C

11. D
12. D
13. B
14. A
15. C

16. D
17. B
18. A
19. A
20. A

21. C
22. C
23. B
24. A
25. C

26. B
27. B
28. D
29. D
30. B

Solutions:

1. **A:** By the distance formula, this is $(3 - (-2))^2 + (-6 - 1)^2 = \boxed{74}$.
2. **B:** Substituting the second equation into the first gives $y = 2(y - 2) + 1 \rightarrow y = 2y - 4 + 1 \rightarrow y = 3$. Plugging this into the first equation gives $x = 3 - 2 = 1$, so $(a, b) = (3, 1)$, so $a + b = \boxed{4}$.
3. **B:** The given line has slope 2, so the perpendicular line has slope $-\frac{1}{2}$. Plugging in the given point gives $x + 2y = \boxed{13}$.
4. **D:** Expanding gives $x^2 - y^2 = 1$, which is a hyperbola.
5. **D:** C is $\frac{2}{3}$ of the way from $(1, 2)$ to $(10, -10)$. Going coordinate by coordinate gives the x -coordinate being $1 + \frac{2}{3}(10 - 1) = 7$ and the y -coordinate $2 + \frac{2}{3}(-10 - 2) = -6$, so $C = \boxed{(7, -6)}$.
6. **B:** Note that any line that splits a circle into two regions of equal area must pass through the center of the circle (sliding a line away from the center increases the area of one part while decreasing the area of the other). Completing the square gives $(x - 4)^2 + (y - 7)^2 = 65$, so the center is $(4, 7)$. Thus the line is $y = 2x - 1$, so $b = \boxed{-1}$.
7. **C:** The set of points equidistant from $(6, 7)$ and $(10, 5)$ is the perpendicular bisector. The line between them has slope $\frac{5-7}{10-6} = -\frac{1}{2}$ and midpoint $(\frac{6+10}{2}, \frac{7+5}{2}) = (8, 6)$. The perpendicular line has slope 2 and thus equation $2x - y = 10$, and setting $y = 0$ gives an x -intercept of $(\boxed{5}, 0)$.
8. **A:** Factor to get $r = \cos(\theta) \sin(\theta) (\cos^2(\theta) + \sin^2(\theta))(\cos^2(\theta) - \sin^2(\theta))$. Using double angle and the Pythagorean identity gives $r = \frac{1}{2} \sin(2\theta) \cos(2\theta)$, and double angle once more gives $\frac{1}{4} \sin(4\theta)$, which is a rose curve with $\boxed{8}$ petals.
9. **D:** Since the diagonals of a parallelogram bisect each other, $A + C = B + D$. Therefore the coordinates of D are $(1 + 0 - (-3), 2 + 5 - 4) = \boxed{(4, 3)}$.
10. **C:** Since the discriminant of a hyperbola is positive, any hyperbola can be factored into the form $(Ax + By + C)(Dx + Ey + F) = G$ for appropriate real numbers A - G . We claim that $Ax + By + C = 0$ and $Dx + Ey + F = 0$ are the asymptotes of the hyperbola.

First, note that since the discriminant is nonzero, these two lines do not have the same slope. Thus, as we let x and/or y get arbitrarily large along the hyperbola, the point gets arbitrarily far from one of the two lines. In order to keep the product constant, the distance to the other line must approach 0, which is precisely what it means to be an asymptote.

Therefore, the hyperbola can be written as $(3x - y - 1)(x - 2y + 4) = G$ for some G . Plugging in $(2, 1)$ gives $G = 4(4) = 16$. Then plugging in $x = 2$ gives $(5 - y)(6 - 2y) = 16$, which expands to $2y^2 - 16y + 14 = 0 \rightarrow (y - 7)(y - 1) = 0$, so the other intersection point is $\boxed{(2, 7)}$.

11. D: Let $O = (0,0)$, $A = \left(2, \frac{5\pi}{12}\right)$, and $B = \left(-2, \frac{\pi}{12}\right) = \left(2, \frac{13\pi}{12}\right)$. Note that $OA = OB = 2$, so $\triangle OAB$ is isosceles, and we can compute $\angle AOB = \frac{2\pi}{3} = 120^\circ$. Letting X be the midpoint of $\overline{AB}$, $\overline{OX}$ is a median, altitude, and angle bisector, splitting $\triangle OAB$ into two 30-60-90 triangles with $OX = 1$, and since $\overline{OX}$ is an angle bisector, the argument is $\frac{\frac{5\pi}{12} + \frac{13\pi}{12}}{2} = \frac{3\pi}{4}$. Therefore the desired midpoint is $\boxed{\left(1, \frac{3\pi}{4}\right)}$.

12. D: We need to find $t_1 \neq t_2$ such that $\cos(3t_1) = \cos(3t_2)$ and $\cos(4t_1) = \cos(4t_2)$. The former implies that $3t_1 = 3t_2 + 2k_1\pi$ or $3t_1 = 2\pi k_1 - 3t_2$ for some k_1 , while the latter implies that $4t_1 = 4t_2 + 2k_2\pi$ or $4t_1 = 2\pi k_2 - 4t_2$ for some k_2 . Re-normalize to get

$$\left(t_1 = t_2 + \frac{2k_1\pi}{3}\right) \text{ or } \left(t_1 = \frac{2k_1\pi}{3} - t_2\right) \text{ and } \left(t_1 = t_2 + \frac{k_2\pi}{2}\right) \text{ or } \left(t_1 = \frac{k_2\pi}{2} - t_2\right).$$

Staring at the graph, we can trace from top right $\rightarrow$ bottom left $\rightarrow$ top $\rightarrow$ bottom right $\rightarrow$ top left. The first quadrant intersection is the one in the top right, where $0 < t_1 < \frac{\pi}{8}$ ($t = \frac{\pi}{8}$ is when the graph hits the y -axis), while $\frac{\pi}{2} < t_2 < \frac{2\pi}{3}$ (between the graph hitting the y -axis again at the top and the graph hitting $x = 1$).

These bounds rule out the first equation, so $t_1 = \frac{2k_2\pi}{3} - t_2$, specifically where $k_2 = 1$. The bounds also rule out the last equation ($k_2 = 1, 2$ are the only reasonable candidates, and neither works), so $t_1 = t_2 + \frac{k_2\pi}{2}$, specifically for $k_2 = -1$. This gives the system of equations $t_1 + t_2 = \frac{2\pi}{3}$ and $t_1 - t_2 = -\frac{\pi}{2}$, which solves to $(t_1, t_2) = \left(\frac{\pi}{12}, \frac{7\pi}{12}\right)$, which gives the point $\boxed{\left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right)}$.

13. B: Translate $(1,2)$ to $(0,0)$, so $(4,3)$ becomes $(3,1)$. We then want to rotate $(3,1)$ by 60° about the origin. Following the hint, and recalling that a rotation by 60° is equivalent to multiplying by $\text{cis}(60^\circ)$ in the complex plane, we can compute

$$(3 + i) \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) = \left(\frac{3 - \sqrt{3}}{2}, \frac{1 + 3\sqrt{3}}{2} \right).$$

Translating back (adding $(1,2)$) gives a final answer of $\boxed{\left(\frac{5}{2} - \frac{\sqrt{3}}{2}, \frac{5}{2} + \frac{3\sqrt{3}}{2}\right)}$.

14. A: Inspired by the previous problem, drop into the complex plane with $a = 0$, $b = 3$, $c = 4i$, and $p = 1$. Q lying on $\overline{AC}$ means $q = xi$ for some real x . Note that PQR being equilateral is equivalent to R being the rotation of Q by 60° clockwise about P (which then has to lie on $\overline{BC}$, which is the line $4x + 3y = 12$). Doing what we did in the previous problem, we get

$$r = (xi - 1) \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) + 1 = \left(\frac{\sqrt{3}}{2}x + \frac{1}{2} \right) + \left(\frac{1}{2}x + \frac{\sqrt{3}}{2} \right)i$$

Plugging this into the line equation gives

$$\left(2\sqrt{3} + \frac{3}{2}\right)x + \left(2 + \frac{3\sqrt{3}}{2}\right) = 12 \rightarrow x = \frac{20 - 3\sqrt{3}}{4\sqrt{3} + 3} = \boxed{\frac{89\sqrt{3} - 96}{39}}$$

15. C: Notice that this is the largest value of k such that the line $y = k$ intersects the rotated ellipse. Rotating *clockwise* by $\frac{\pi}{4}$ gives back the original ellipse, and $y = k$ becomes $\frac{x+y}{\sqrt{2}} = k$ (one way to think about this is that the slope needs to be -1 , and it represents a distance of k from the origin in the same way $y = k$ does).

Letting $\ell = \sqrt{2}k$ for convenience, we are interested in the maximum ℓ such that $x + y = \ell$ intersects $\frac{x^2}{4} + y^2 = 1$. Substituting in $y = \ell - x$ gives $\frac{x^2}{4} + x^2 - 2\ell x + \ell^2 = 1$. This equation has either zero, one, or two solutions for x , and the maximum value of ℓ will occur when there is only one solution (you can think about the line $x + y = \ell$ being tangent to the ellipse).

Taking the discriminant (with respect to x) gives $(-2\ell)^2 - 4\left(\frac{5}{4}\right)(\ell^2 - 1) = 5 - \ell^2$, which is zero when $\ell = \pm\sqrt{5}$. The positive one corresponds to the max, so substituting back $k = \frac{\ell}{\sqrt{2}}$ gives $\boxed{\frac{\sqrt{10}}{2}}$.

16. D: Add all three vectors to get $2(\vec{u} + \vec{v} + \vec{w}) = \langle 3 - 2 + 5, 6 + 1 - 3 \rangle = \langle 6, 4 \rangle$. Dividing by 2 gives $\boxed{\langle 3, 2 \rangle}$.

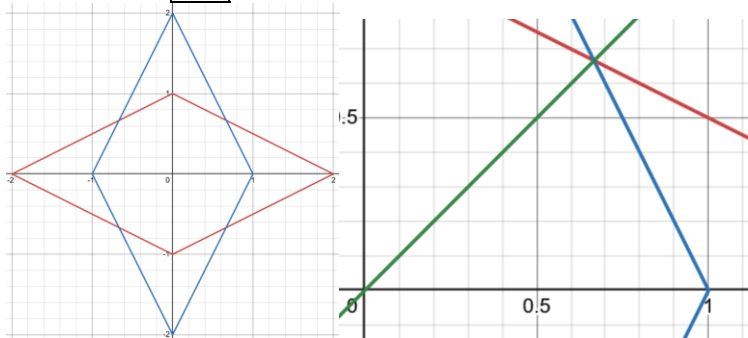
17. B: The polar form of a conic section is $r = \frac{ep}{1 + e \cos(\theta)}$, where e is the eccentricity and p is the distance from the focus at the pole to its corresponding directrix. Dividing the numerator and denominator by 3 gives $\frac{\frac{4}{3}}{1 + \frac{2}{3} \cos(\theta)}$, so $e = \frac{2}{3} < 1$, and thus the conic is a Non – Circular Ellipse.

18. A: Recall that the given locus is an ellipse with major axis $2a = 14$. The distance between the foci is $2c = \sqrt{4^2 + 6^2} = 2\sqrt{13}$. It follows that the semiminor axis is $b = \sqrt{a^2 - c^2} = \sqrt{49 - 12} = 6$. It follows that the area is $\pi(7)(6) = \boxed{42\pi}$.

19. A: Note that $-4 = \vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos(\theta) = 12 \cos(\theta)$, so $\cos(\theta) = -\frac{1}{3}$. It follows that $\sin(\theta) = \sqrt{1 - \left(-\frac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3}$, so $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \|\vec{w}\| \sin(\theta) = 12 \left(\frac{2\sqrt{2}}{3}\right) = \boxed{8\sqrt{2}}$.

20. A: Considering the first quadrant, $x + 2y = 2$ and $2x + y = 2$. Reflecting over the x and y axes (absolute values give symmetry in each) give rhombuses centered at the origin with diagonals 2 and 4, like the diagram below.

The $\leq$'s mean we are interested in the intersection of the interiors of these rhombuses, which is an octagon. Note that $x + 2y = 2$ and $2x + y = 2$ intersect at $(2/3, 2/3)$. From here note that the axes and the lines $y = \pm x$ split the octagon into eight congruent triangles. One of these triangles has base 1 and height $\frac{2}{3}$, for an area of $1/3$. This gives a total area of $\boxed{8/3}$.



21. C: Using the usual naming convention, $a = 3$, $b = 4$, so $c = \sqrt{3^2 + 4^2} = 5$, and the focus is $(5,0)$. We want the intersection of this hyperbola with the line $x = 5$, so we can plug in $\frac{25}{9} - \frac{y^2}{16} = 1$. Solving gives $\frac{16}{9} = \frac{y^2}{16}$, or $y = \pm \frac{16}{3}$. This gives a distance of $\boxed{\frac{32}{3}}$.

Remark: The line segment connecting these two points is a latus rectum of the hyperbola, which more generally has length $\frac{2b^2}{a}$.

22. C: Note that the distance from the vertex to the focus is 5, so the distance from the vertex to the directrix is also 5. If we draw the line connecting the vertex V and the focus F , this line meets the directrix at the point D such that V is the midpoint of $\overline{FD}$. It follows that $D = (-6, -8)$. Furthermore, the directrix is perpendicular to the line connecting V and F , which has slope $\frac{4}{3}$. It follows that the directrix has slope $-\frac{3}{4}$ and thus equation $3x + 4y = -50$.

We then have that the parabola is the set of all points equidistant from the point $(0,0)$ and the line $3x + 4y + 50 = 0$. This gives the equation

$$\sqrt{x^2 + y^2} = \frac{|3x + 4y + 50|}{\sqrt{3^2 + 4^2}}.$$

Clearing denominators and squaring both sides gives

$$25x^2 + 25y^2 = 9x^2 + 16y^2 + 24xy + 300x + 400y + 2500.$$

Setting $x = 0$ and rearranging gives $9y^2 - 400y - 2500 = 0$. This factors to $(9y + 50)(y - 50) = 0$, which gives the positive solution $y = \boxed{50}$.

23. B: We can compute $A + C = 29$ and $B^2 - 4AC = -144$. Using the invariant and noting that $B' = 0$, we get $A' + C' = 29$ and $-4A'C' = -144 \rightarrow A'C' = 36$. Squaring the desired quantity gives $A' + C' + 2\sqrt{A'C'} = 29 + 2(6) = 41$, so $\sqrt{A'} + \sqrt{C'} = \boxed{\sqrt{41}}$.

24. A: Following the hint, we can put in standard form $\frac{x^2}{(\sqrt{F'/A'})^2} + \frac{y^2}{(\sqrt{F'/C'})^2} = 1$, which gives an area of $\frac{\pi F'}{\sqrt{A'C'}}$.

Since $B' = 0$, we once again get $A'C' = 36$ and thus $\frac{2\pi}{6} = \boxed{\frac{\pi}{3}}$.

25. C: Recall (or rederive...) that $x = x' \cos(\theta) - y' \sin(\theta)$ and $y = x' \sin(\theta) + y' \cos(\theta)$. Expanding out the degree 1 term gives $D(x' \cos(\theta) - y' \sin(\theta)) + E(x' \sin(\theta) + y' \cos(\theta)) = (D \cos(\theta) + E \sin(\theta))x' + (-D \sin(\theta) + E \cos(\theta))y'$, so $D' = D \cos(\theta) + E \sin(\theta)$ and $E' = -D \sin(\theta) + E \cos(\theta)$. Wishful thinking with the identity $\sin^2(\theta) + \cos^2(\theta) = 1$ suggests to try $D^2 + E^2$, and indeed

$$\begin{aligned} D'^2 + E'^2 &= (D \cos(\theta) + E \sin(\theta))^2 + (-D \sin(\theta) + E \cos(\theta))^2 \\ &= (D^2 + E^2)(\cos^2(\theta) + \sin^2(\theta)) + (2 - 2)DE \sin(\theta) \cos(\theta) \\ &= \boxed{D^2 + E^2}. \end{aligned}$$

26. B: $r = 2 \cos(4\theta)$ is a rose curve with 8 petals, while $r = 1$ is a circle. The circle intersects each petal twice, for a total of $\boxed{16}$ intersections.

27. B: Write $z = re^{i\theta}$. Then $z^2 = r^2 e^{2i\theta}$ and $z^3 = r^3 e^{3i\theta}$. Then the triangle formed by 0, z , and z^3 has area $\frac{1}{2}r(r^3) \sin(2\theta)$, while the triangle formed by 0, z , and z^2 has area $\frac{1}{2}r(r^2) \sin(\theta)$. Dividing and using double angle gives $2r \cos(\theta)$, which we're told is equal to 4. But $r \cos(\theta)$ is equal to $\operatorname{Re}(z)$, so our answer is $\boxed{2}$.

28. D: Following the hint, we first project onto the xy -plane, and use shoelace. Writing the points in counterclockwise order gives $(1, -3), (2, 2), (-3, 1), (-4, -2), (0, -4)$, then we compute

$$\frac{|(1(2) + 2(1) + (-3)(-2) + (-4)(-4) + 0(-3)) - (-3(2) + 2(-3) + 1(-4) + (-2)(0) + (-4)(1))|}{2}$$

$$= \frac{|26 - (-20)|}{2} = 23.$$

The key observation is that “un-projecting” onto $x + y + 2z = 0$ is a linear transformation, so doing this un-projection scales *any* shape by the same area! (This scale factor is the determinant of some matrix we’re not going to compute).

With this idea in hand, let’s pick an easier shape to think about, say the right triangle with vertices $(0,0), (1, -1)$, and $(1,1)$. When we un-project onto the plane, we get $(0,0,0)$ and $(1, -1,0)$, so one of the legs stays unchanged, while the other vertex becomes $(1,1, -1)$, which means the leg between it and $(0,0)$ gets scaled up from $\sqrt{2}$ to $\sqrt{3}$. The triangle remains a right triangle, so the ratio of areas is simply $\frac{\frac{1}{2}\sqrt{2}\cdot\sqrt{3}}{\frac{1}{2}\sqrt{2}\cdot\sqrt{2}} = \frac{\sqrt{6}}{2}$. It follows that the final area is $\boxed{\frac{23\sqrt{6}}{2}}$.

29. D: Let d be the distance from the plane to the center $(0,0,0)$ of the sphere. Then the radius of the circle of intersection is $\sqrt{4^2 - d^2}$ by the Pythagorean theorem. Using the point to plane formula gives $d = \frac{|1(0)+1(0)+1(0)-3|}{\sqrt{1^2+1^2+1^2}} = \sqrt{3}$, so the radius of the circle is $\sqrt{13}$, giving an area of $\boxed{13\pi}$.

30. B: We combine the ideas of the previous two problems: transform the ellipsoid into a sphere and keep track of how the area of the ellipse changes which each step.

First, we scale by $\frac{1}{2}$ in the y direction by replacing y with $2y$, getting $x + 2y + z = 0$ and $x^2 + y^2 + \frac{z^2}{4} = 1$. Then we scale by $\frac{1}{4}$ in the z direction by replacing z with $4z$ getting $x + 2y + 4z = 0$ and $x^2 + y^2 + z^2 = 1$. This is a sphere of radius 1 and a plane passing through its center so the intersection is a circle with radius 1 and area π .

Now we go backwards, tracking how this circle gets scaled by each transformation by noting the linear transformation and considering a right triangle that’s easier to think about. Consider the points $(0,0,0), (-2,1,0)$, and $(4,8,-5)$ on $x + 2y + 4z = 0$. When scaling z by 4 to get $x + 2y + z = 0$, the first two points stay the same while the third becomes $(4,8,-20)$. This corresponds to a ratio of lengths of $\sqrt{\frac{480}{105}} = \frac{4\sqrt{2}}{\sqrt{7}}$.

Doing it again, consider the points $(0,0,0), (1,0,-1)$, and $(1,-1,1)$ on $x + 2y + z = 0$. When scaling y by 2 to get $x + y + z = 0$, the first two points stay the same, while the third becomes $(1,-2,1)$. This corresponds to a ratio of lengths of $\frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$. Composing both gives a multiplication by $\frac{8}{\sqrt{7}}$, so our final area is $\boxed{\frac{8\pi}{\sqrt{7}}}$.

If you’re curious what this looks like, see <https://www.desmos.com/3d/bpg4xrrb3z> (try modifying the equations as described above!)