

Answers:

1. B
2. E
3. B
4. D
5. C
6. A
7. D
8. B
9. A
10. C
11. E
12. B
13. A
14. D
15. C
16. D
17. C
18. A
19. B
20. C
21. D
22. E
23. A
24. B
25. D
26. C
27. C
28. C
29. D
30. B

Solutions:

1. The possibilities here are TH, TTTH, TTTTH, and any odd number of tails followed by a heads. Since T and H are equally likely each time and the flips are independent, the probability

is the geometric series $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^6 + \dots$, which has a sum of $\frac{\left(\frac{1}{2}\right)^2}{1 - \left(\frac{1}{2}\right)^2} = \frac{1}{3}$.

2. The train leaving from Dallas covers 60 miles before the train leaving from Texarkana leaves, so the distance between the trains when the train leaving from Texarkana leaves has been reduced to 120 miles. Since the trains travel toward each other, their effective rate of travel is 150 miles per hour, so it would take another $120 \cdot \frac{1}{150} = \frac{4}{5}$ hours for the trains to meet, which is 48 minutes. Therefore, the trains meet at 3:48 pm.

3. After each year, $\frac{19}{20}$ of the substance remains. Therefore, after three years, there is

$200\left(\frac{19}{20}\right)^3 = \frac{6859}{40} = 171.475$ grams, so to the nearest gram, this is 171 grams.

4. At that point, the car traveling north would have traveled for four hours, or 160 miles. The car traveling east would have traveled for two hours, or 120 miles. Since the cars are traveling due north and due east, there is a right angle between them, so the two legs of the right triangle have lengths 120 miles and 160 miles, and the hypotenuse would thus have a length of 200 miles (this is a 3-4-5 right triangle scaled up by a factor of 40).

5. The value, in dollars, of the investment would be $5000\left(1 + \frac{.04}{4}\right)^{4 \cdot 1} = 5000(1.01)^4$
 $= 5000(1.04060401) = 5203.02005$, so to the nearest dollar, this would be \$5203.

6. The drawn marbles could both be red, both be blue, or both be green. Therefore, the

probability is $\frac{\binom{5}{2} + \binom{4}{2} + \binom{3}{2}}{\binom{12}{2}} = \frac{10 + 6 + 3}{66} = \frac{19}{66}$.

7. The drawn marbles could be two red with either a blue or green, two blue with either a red or green, or two green with either a red or blue. Therefore, the probability that this occurs is

$$\frac{\binom{6}{2}\left(\binom{5}{1}+\binom{4}{1}\right)+\binom{5}{2}\left(\binom{6}{1}+\binom{4}{1}\right)+\binom{4}{2}\left(\binom{6}{1}+\binom{5}{1}\right)}{\binom{15}{3}} = \frac{15 \cdot 9 + 10 \cdot 10 + 6 \cdot 11}{455} = \frac{301}{455} = \frac{43}{65}.$$

8. Modeling the parabola with its vertex at the origin makes the equation of the parabola $x^2 = 8y$. Now, if the opening is 6 m wide, the points at the edge of the opening have x -coordinates ± 3 . Plugging either of those in gives the height above the vertex for those points as $\frac{9}{8} = 1.125$, which would also then be the depth of the dish.

9. The water level will rise uniformly, so just take the rate of increase of the volume of water and divide it by the area of the base: $\frac{2}{\pi \cdot 3^2} = \frac{2}{9\pi}$.

10. Billy's current average is $\frac{82+88+91+94}{4} = \frac{355}{4} = 88.75$, so he must score better than his lowest score of 82 in order to bring up his average. Since an average of 90 would equal $\frac{360}{4}$, Billy needs to bring his total points up by 5. Therefore, replacing his 82 with an 87 should do the trick. Just for confirmation, $\frac{87+88+91+94}{4} = \frac{360}{4} = 90$.

11. Let x be the number of \$1 decreases in price that are made. Then based on the information provided, if the business sells $200 + 20x$ items, the price per item is $10 - x$ dollars, making the revenue $(200 + 20x)(10 - x) = 2000 - 20x^2$, which is clearly maximized when no \$1 decreases are made. Therefore, the revenue is maximized at the \$10 price.

12. $1.08^x = 2 \Rightarrow x = \log_{1.08} 2$, so you could try to figure out the nearest whole number that solves this. However, you could also use the rule of 72, which says that 72 divided by the interest rate is approximately equal to the number of years it will take to double. $\frac{72}{8} = 9$, so this should be close to 9 years (and, $\log_{1.08} 2 \approx 9.006$, confirming this answer).

13. $\frac{s}{t} = \frac{2\pi \cdot 15 \text{ meters}}{3 \text{ minutes}} \cdot \frac{1 \text{ minute}}{60 \text{ seconds}} = \frac{\pi}{6} \frac{\text{meters}}{\text{seconds}}$

14. The highest and lowest points of a cart are 3 and 33 meters off the ground, so the amplitude is 15 meters and the midline is 18 meters. The period of the wheel is 3, so the coefficient of t is $\frac{2\pi}{3}$, and of the expressions that fit this description thus far, the only one that equals 3 when $t=0$ is choice D.

15. For rows 30 through 5, the price per seat is every whole dollar amount from 50 to 75; then, the price per seat in the first four rows is 74, 73, 72, and 71. Therefore, the total price for one seat in every single row is $(50 + 51 + \dots + 75) + (74 + 73 + 72 + 71) = \frac{26}{2}(50 + 75) + 290 = 1915$. Since there are 20 seats per row, the total revenue is $1915 \cdot 20 = 38,300$ dollars.

16. 850 hours is 1.5 standard deviations below the mean. According to the empirical rule, scores at 1 standard deviation are in the 16th percentile, and scores at 2 standard deviations are in the 2.5th percentile. If the decline between these was linear, the halfway score of 1.5 standard deviations would be at the 9.25th percentile, but a normal distribution drops more quickly than that, so the only reasonable answers are D and E. However, E is almost the full 2 standard deviations below, so that is not reasonable. The closest value would be 7% (and the actual answer is about 6.7%).

17. The distance from $(2,0)$ to (x,y) on that graph is $\sqrt{(x-2)^2 + (y-0)^2} = \sqrt{(x-2)^2 + (x+4)^2}$
 $= \sqrt{x^2 - 4x + 4 + x + 4} = \sqrt{x^2 - 3x + 8} = \sqrt{\left(x - \frac{3}{2}\right)^2 + \frac{23}{4}}$, so the minimum distance is $\sqrt{\frac{23}{4}}$
 $= \frac{\sqrt{23}}{2}$.

18. The ways to get a sum of 16 are 1,6,6 (3 ways); 2,5,6 (6 ways); 3,4,6 (6 ways); 3,5,5 (3 ways); and 4,4,5 (3 ways). Therefore, the probability is $\frac{3+6+6+3+3}{6^3} = \frac{21}{216} = \frac{7}{72}$.

19. There are $5^3 = 125$ ways to not roll at least one six, meaning there are $216 - 125 = 91$ rolls with at least one six. From the previous problem, there are $3+6+6 = 15$ ways among these with at sum of 13, so the probability is $\frac{15}{91}$.

20. Since the probability of rolling a sum of 13 is $\frac{7}{72}$, the expected number of rolls before getting that is $\frac{72}{7} = 10\frac{2}{7}$, so the nearest integer to this is 10.

21. Let y be the length of fencing running parallel to the barn, and let x be the lengths of fencing running perpendicular to the barn. Using all of the fencing would maximize the enclosed area, so $2x + y = 120 \Rightarrow y = 120 - 2x$. Therefore, the area enclosed by the pen is $xy = x(120 - 2x) = 120x - 2x^2 = 1800 - 2(x - 30)^2$, so the maximum area of the pen is 1800 square meters.

22. The plant should be placed at the circumcenter of the triangle whose vertices are at the cities. The semi-perimeter of this triangle is $\frac{13+14+15}{2} = 21$, so using Hero's formula, the area enclosed by the triangle is $\sqrt{21(21-13)(21-14)(21-15)} = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$. Therefore, the radius of the circumscribed circle of this triangle is $R = \frac{13 \cdot 14 \cdot 15}{4 \cdot 84} = \frac{65}{8}$. Therefore, we would need 3 times this for the pipeline, one radius in the direction of each of the three cities. This total length is $3 \cdot \frac{65}{8} = \frac{195}{8} = 24.375$ miles.

23. Model Charybdis as a disk centered at the origin with radius 4. Essentially we only need the boundary circle of this disk, so Charybdis is $a^2 + b^2 = 16$, where (a, b) is any point on the right side of this circle. Therefore, model Scylla as the point $(6, 0)$. Then Odysseus's path is along points (x, y) , where $(x - 6)^2 + y^2 = (x - a)^2 + (y - b)^2 \Rightarrow x^2 - 12x + 36 + y^2 = x^2 - 2ax + a^2 + y^2 - 2by + b^2 = x^2 - 2ax + y^2 - 2by + 16 \Rightarrow -12x + 20 = -2ax - 2by \Rightarrow 6x - 10 = ax + by \Rightarrow (6 - a)x - 10 = by \Rightarrow (6 - a)^2 x^2 - 20(6 - a)x + 100 = b^2 y^2 = (16 - a^2)y^2 \Rightarrow (6 - a)^2 x^2 - (16 - a^2)y^2 - 20(6 - a)x + 100 = 0$. Now, passing between these objects, we must have $x > 0$, so since $0 \leq a \leq 4$, we break this up into two scenarios: 1) $a = 4$, and 2) $0 \leq a < 4$. If $a = 4$, $b = 0$, so $x = 5 \Rightarrow y = 0$, so this is just the point $(5, 0)$. If $0 \leq a < 4$, we have an equation that looks like a conic section, so check its discriminant: $0^2 - 4(6 - a)^2(-(16 - a^2)) = 4(6 - a)^2(16 - a^2)$, which for these values must be positive, so this is a hyperbola.

$$24. \frac{\binom{15}{2} \binom{5}{2}}{\binom{20}{4}} = \frac{\frac{15 \cdot 14}{2 \cdot 1} \cdot \frac{5 \cdot 4}{2 \cdot 1}}{\frac{20 \cdot 19 \cdot 18 \cdot 17}{4 \cdot 3 \cdot 2 \cdot 1}} = \frac{70}{323}$$

25. The probability of choosing all five defective machines is $\frac{\binom{5}{5}}{\binom{20}{5}} = \frac{1}{\binom{20}{5}}$, so the expected

number of times it will take to pull all five defective machines is $\binom{20}{5} = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 3}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 15,504$.

26. In the worst case scenario, the inspector selects the 15 functional machines first, so she must choose two more to guarantee she has selected two defective ones, for a total of 17 machines.

27. The circumference of the new circle traced by the center of the smaller circle is $2\pi(R+r)$, and the circumference of the smaller circle is $2\pi r$, so the number of times the smaller circumference will lay along the larger circumference is $\frac{2\pi(R+r)}{2\pi r} = \frac{R+r}{r}$.

28. By one of Vieta's formulas, the product of the zeros of an odd-degree polynomial would equal the negative of the constant term divided by the leading coefficient of 1, i.e., the negative of the constant term. The product of the roots is 1 (there are two pairs of trigonometric functions that are reciprocals of each other), so the constant term of the polynomial is equal to -1 —which is, of course, the value of $p(0)$.

29. Let x be the distance between the platform and the stilt walker, and let z be the height of the stilt walker for which we are looking. Because the angle of depression to the top of the stilt walker is 30° , we have the height from the tightrope walker's eyes down to the top of the stilt walker is $\frac{x}{\sqrt{3}} = \frac{x\sqrt{3}}{3}$. Therefore, $\frac{x\sqrt{3}}{3} + z = 35$. Additionally, because the angle of depression

to the floor is 75° , $\frac{35}{x} = \tan 75^\circ = 2 + \sqrt{3} \Rightarrow x = \frac{35}{2 + \sqrt{3}} = 35(2 - \sqrt{3})$. Therefore, $\frac{35(2 - \sqrt{3})\sqrt{3}}{3}$

$$+z = 35 \Rightarrow 70\sqrt{3} - 105 + 3z = 105 \Rightarrow 3z = 210 - 70\sqrt{3} \Rightarrow z = \frac{70(3 - \sqrt{3})}{3}.$$

30. For intersections using polar equations, one must be careful. It is easy to say that for this problem, we must have $5\cos\theta = 2 + 2\cos\theta \Rightarrow 3\cos\theta = 2 \Rightarrow \cos\theta = \frac{2}{3}$, and that there are two angles in the interval $[0, 2\pi]$ that satisfy this equation. However, remember that $r = 5\cos\theta$ traces itself twice on that interval, and remember that $r = 0$ can be covered by any angle. $\cos\theta = \frac{2}{3}$ does give the same r -value in both equations, so that is two points (one in quadrant I, one in quadrant IV). Further, $r = 0$, representing the origin, is achievable in both equations. Lastly, $r = 2 + 2\cos\theta$ cannot be negative, so there is no concern about negative values of r . Hence, there are three intersection points.

