

Answers:

1. B
2. A
3. A
4. B
5. C
6. B
7. D
8. C
9. E
10. A
11. D
12. C
13. D
14. E
15. C
16. D
17. B
18. A
19. B
20. C
21. A
22. D
23. C
24. C
25. A
26. D
27. B
28. E
29. E
30. B

Solutions:

$$1. 2x - 7 = 5 \Rightarrow 2x = 12 \Rightarrow x = 6$$

$$2. 0 = x^2 - 9 = (x - 3)(x + 3) \Rightarrow x = \pm 3$$

$$3. \frac{3}{x} = 6 \Rightarrow x = \frac{3}{6} = \frac{1}{2}$$

$$4. |x - 4| < 3 \Rightarrow -3 < x - 4 < 3 \Rightarrow 1 < x < 7$$

$$5. 2^x = 16 \Rightarrow x = 4$$

$$6. 3 = \log_2(x + 1) + \log_2(x - 1) = \log_2((x + 1)(x - 1)) = \log_2(x^2 - 1) \Rightarrow x^2 - 1 = 2^3 = 8 \Rightarrow x^2 = 9 \\ \Rightarrow x = 3 \text{ (because } x = -3 \text{ makes the two logarithm arguments negative)}$$

$$7. 2\sin x \cos x = \sin(2x) = \sin x \Rightarrow \sin x = 0 \text{ or } \cos x = \frac{1}{2} \Rightarrow x = 0, \pi, \frac{\pi}{3}, \frac{5\pi}{3} \Rightarrow 0 + \pi + \frac{\pi}{3} + \frac{5\pi}{3} = 3\pi$$

$$8. 0 < \frac{x^2 - 9}{x - 2} = \frac{(x - 3)(x + 3)}{x - 2}, \text{ and using sign analysis, all three factors are positive if } x > 3, \text{ while } x - 3 \text{ and } x - 2 \text{ are negative while } x + 3 \text{ is positive for } -3 < x < 2; \text{ these are the only ways to get the expression to be positive.}$$

$$9. 2^{x+2} - 2^{x-1} = 2^x(2^2 - 2^{-1}) = 5\left(4 - \frac{1}{2}\right) = 5\left(\frac{7}{2}\right) = \frac{35}{2}$$

$$10. 13 = x^2 + y^2 = (x + y)^2 - 2xy = 5^2 - 2xy = 25 - 2xy \Rightarrow 2xy = 12 \Rightarrow xy = 6, \text{ so } x \text{ and } y \text{ are the solutions to the equation } 0 = z^2 - 5z + 6 = (z - 2)(z - 3) \Rightarrow z = 2 \text{ or } z = 3. \text{ Since } x < y, x = 2 \text{ and } y = 3.$$

$$11. \text{ Let } a \text{ and } b \text{ be the two solutions to the original equation. Then } a + b = \frac{7}{3} \text{ and } ab = \frac{2}{3}, \text{ so}$$

$$a^2 + b^2 = (a + b)^2 - 2ab = \left(\frac{7}{3}\right)^2 - 2\left(\frac{2}{3}\right) = \frac{49}{9} - \frac{4}{3} = \frac{49}{9} - \frac{12}{9} = \frac{37}{9} \text{ and } a^2b^2 = (ab)^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}.$$

Therefore, the squares of the solutions satisfy the equation $x^2 - \frac{37}{9}x + \frac{4}{9} = 0$, or

$$9x^2 - 37x + 4 = 0.$$

12. $\log_3(x^2 - 1) = 2 \Rightarrow x^2 - 1 = 3^2 = 9 \Rightarrow x^2 = 10 \Rightarrow x = \pm\sqrt{10}$ (since the logarithm's argument is positive for both values)

13. $\cos(2x) = \frac{1}{2} \Rightarrow 2x = \frac{\pi}{3} + 2\pi k$ or $2x = \frac{5\pi}{3} + 2\pi k \Rightarrow x = \frac{\pi}{6} + \pi k$ or $x = \frac{5\pi}{6} + \pi k$. This makes the solutions $\frac{\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{6},$ and $\frac{11\pi}{6}$, and $\frac{\pi}{6} + \frac{7\pi}{6} + \frac{5\pi}{6} + \frac{11\pi}{6} = \frac{24\pi}{6} = 4\pi$.

14. $0 \geq \frac{x^2 - 4}{x + 1} = \frac{(x - 2)(x + 2)}{x + 1}$, and using sign analysis, all three factors are negative if $x < -2$, while $x + 1$ and $x + 2$ are positive while $x - 3$ is negative for $-1 < x < 2$; these are the only ways to get the expression to be negative. Throw in ± 2 since they make the expression equal to 0, and the final answer is $(-\infty, -2] \cup (-1, 2]$.

15. Let a and b be the two solutions to the original equation, with a being the greater solution.

Then $a + b = k$, $ab = k$, and $a - b = 2$, so from the first and last equations, $a = \frac{k + 2}{2}$ and

$b = \frac{k - 2}{2}$. Hence, $k = \left(\frac{k + 2}{2}\right)\left(\frac{k - 2}{2}\right) = \frac{k^2 - 4}{4} \Rightarrow k^2 - 4 = 4k \Rightarrow k^2 - 4k - 4 = 0 \Rightarrow 8 = k^2 - 4k + 4 = (k - 2)^2 \Rightarrow \pm 2\sqrt{2} = k - 2 \Rightarrow k = 2 \pm 2\sqrt{2}$. However, since the roots are both positive, k (the sum and product of the roots) must also be positive. Therefore, $k = 2 + 2\sqrt{2}$.

16. $n = 3^x - 3^{x-1} - 3^{x-2} = 3^x(1 - 3^{-1} - 3^{-2}) = 3^x\left(1 - \frac{1}{3} - \frac{1}{9}\right) = 3^x\left(\frac{9}{9} - \frac{3}{9} - \frac{1}{9}\right) = 3^x\left(\frac{5}{9}\right) \Rightarrow 3^x = \frac{9}{5}n$, so

for there to be an integral solution to this equation, $n = 5 \cdot 3^m$ for some non-negative integer m . $5 \cdot 3^5 = 5 \cdot 243 = 1215 < 2026$ and $5 \cdot 3^6 = 5 \cdot 729 = 3645 > 2026$, so 3645 is the least such value.

17. The reciprocals are the zeros of the polynomial with the coefficients in reverse order.

Hence, they are the zeros to $-20x^3 + 5x^2 - 7x + 2 = 0$, and the sum of those solutions is

$$-\frac{5}{-20} = \frac{1}{4}.$$

18. For $\pi < \theta < \frac{3\pi}{2}$, $\tan \theta = \frac{3}{4} \Rightarrow \cos \theta = -\frac{4}{5}$, so using the identity $\cos(3\theta) = 4\cos^3 \theta - 3\cos \theta$,

$$\cos(3\theta) = 4\left(-\frac{4}{5}\right)^3 - 3\left(-\frac{4}{5}\right) = 4\left(-\frac{64}{125}\right) + \frac{12}{5} = -\frac{256}{125} + \frac{300}{125} = \frac{44}{125}.$$

19. $0 = x^4 - 13x^2 + 36 = (x^2 - 4)(x^2 - 9) = (x+2)(x-2)(x+3)(x-3)$, so the sum of the positive solutions is $2+3=5$.

20. Multiplying both sides of the equation by $(x-1)(x+2)$ gives $2(x+2)+3(x-1) = (x+2)(x-1) \Rightarrow 2x+4+3x-3 = x^2+x-2 \Rightarrow 0 = x^2-4x-3 \Rightarrow 7 = x^2-4x+4 = (x-2)^2 \Rightarrow \pm\sqrt{7} = x-2 \Rightarrow x = 2 \pm \sqrt{7}$, and both of these values make the expression in the equation defined, so the sum of the solutions is $(2+\sqrt{7})+(2-\sqrt{7})=4$.

21. $0 = x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$, so a, b , and c are 1, 2, and 3, and $1^2 + 2^2 + 3^2 = 1+4+9=14$.

22. Let a and b be the two solutions to the original equation. Then $a+b=p$ and $ab=k$, so $a^2+b^2 = (a+b)^2 - 2ab = p^2 - 2k$.

23. $0 = x^4 - 4x^3 + 5x^2 - 4x + 1 = (x^4 - 4x^3 + 6x^2 - 4x + 1) - x^2 = (x-1)^4 - x^2 = ((x-1)^2 - x)((x-1)^2 + x) = (x^2 - 3x + 1)(x^2 - x - 1)$, so the solutions to the first factor set equal to 0 are $x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(1)}}{2(1)} = \frac{3 \pm \sqrt{5}}{2}$ (both of which are positive), and the solutions to the second factor set equal to 0 are $x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ (both of which are imaginary).

24. Let S_n be the sum of the n th powers of the solutions to the equation. Using Newton's sums, $0 = 2S_1 + 3 \cdot 1 \Rightarrow S_1 = -\frac{3}{2}$. This implies that $0 = 2S_2 + 3S_1 + 4 \cdot 2 = 2S_2 - \frac{9}{2} + 8 \Rightarrow S_2 = -\frac{7}{4}$. Further, $0 = 2S_3 + 3S_2 + 4S_1 = 2S_3 - \frac{21}{4} - 6 \Rightarrow S_3 = \frac{45}{8}$, so $0 = 2S_4 + 3S_3 + 4S_2 = 2S_4 + \frac{135}{8} - 7$

$$\Rightarrow S_4 = -\frac{79}{16}. \text{ Lastly, } 0 = 2S_5 + 3S_4 + 4S_3 = 2S_5 - \frac{237}{16} + \frac{45}{2} \Rightarrow S_5 = -\frac{123}{32}.$$

25. $3 = \log_2 x + \log_2 (x-2) = \log_2 (x(x-2)) = \log_2 (x^2 - 2x) \Rightarrow x^2 - 2x = 2^3 = 8 \Rightarrow 0 = x^2 - 2x - 8$
 $= (x-4)(x+2) \Rightarrow x = 4$ (because $x = -2$ makes the two logarithm arguments negative), so the sum of the solutions is 4.

26. Let $f(x) = ax^2 + bx + c$. Then $f(1-x) = a(1-x)^2 + b(1-x) + c = ax^2 - (2a+b)x + (a+b+c)$.
 Therefore, $ax^2 - (2a+b)x + (a+b+c) = 3x^2 + 7x - 11 \Rightarrow a = 3$. Further, $7 = -(2a+b) = -(6+b) \Rightarrow b = -13$. Lastly, $-11 = a+b+c = 3-13+c \Rightarrow c = -1$. Therefore, $f(x) = 3x^2 - 13x - 1$, and the product of the roots is $\frac{-1}{3} = -\frac{1}{3}$.

27. Since $x^2 = |x|^2$, $0 = 3x^2 + 5|x| - 3 \Rightarrow |x| = \frac{-5 \pm \sqrt{5^2 - 4(3)(-3)}}{2(3)} = \frac{-5 \pm \sqrt{61}}{6}$. However,
 $\frac{-5 - \sqrt{61}}{6} < 0$, so there are no solutions to this part of the equation. $\frac{-5 + \sqrt{61}}{6} > 0$, so the solutions to $|x| = \frac{-5 + \sqrt{61}}{6}$ are $\frac{-5 + \sqrt{61}}{6}$ and $-\left(\frac{-5 + \sqrt{61}}{6}\right) = \frac{5 - \sqrt{61}}{6}$. The first solution is positive while the second solution is negative, so the least real solution is $\frac{5 - \sqrt{61}}{6}$.

28. $f(x) = \sqrt{\frac{63 + 86x - 120x^2}{66x^2 + 37x - 133}} = \sqrt{\frac{(6x-7)(-20x-9)}{(6x-7)(11x+19)}}$, so we need to find where this radicand is non-negative. We will use sign analysis, but we can ignore the common factor of $6x-7$ since its zero is not an integer anyway. Both factors $-20x-9$ and $11x+19$ can't simultaneously be negative, but they are both positive when $-\frac{19}{11} < x < -\frac{9}{20}$. The radicand is 0 for $x = -\frac{9}{20}$ as well. The only integer in this interval is -1 , so there is only one integer in the domain.

29. $\log_x (6x^2 - 8x - 3) = 3 \Rightarrow 6x^2 - 8x - 3 = x^3 \Rightarrow 0 = x^3 - 6x^2 + 8x + 3 = (x-3)(x^2 - 3x - 1)$, so the real solutions are 3 and $\frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2(1)} = \frac{3 \pm \sqrt{13}}{2}$. However, $\frac{3 - \sqrt{13}}{2} < 0$, so this

can't be the base of a logarithm. The other two are positive, and since the argument of the logarithm equals x^3 , if $x > 0$, then so is the argument. The sum of the solutions, then, is

$$3 + \frac{3 + \sqrt{13}}{2} = \frac{6}{2} + \frac{3 + \sqrt{13}}{2} = \frac{9 + \sqrt{13}}{2}.$$

30. Let A be the area enclosed by the triangle. Since the length of an altitude of a triangle is twice the area divided by the side length of the side to which the altitude is drawn, the three

side lengths of the triangle are $\frac{2A}{\frac{168}{13}} = \frac{13A}{84}$, $\frac{2A}{12} = \frac{14A}{84}$, and $\frac{2A}{\frac{56}{5}} = \frac{5A}{28} = \frac{15A}{84}$. The semi-

perimeter of the triangle is $\frac{1}{2} \left(\frac{13A}{84} + \frac{14A}{84} + \frac{15A}{84} \right) = \frac{21A}{84}$, so using Hero's formula, the enclosed

area is $A = \sqrt{\frac{21A}{84} \cdot \frac{8A}{84} \cdot \frac{7A}{84} \cdot \frac{6A}{84}} = \frac{84A^2}{84^2} \Rightarrow A = 84$, which makes the semi-perimeter $\frac{21 \cdot 84}{84} = 21$,

making the perimeter $2(21) = 42$.