

Answers:

1. B
2. D
3. B
4. C
5. C
6. A
7. D
8. A
9. D
10. E
11. B
12. A
13. E
14. C
15. D
16. D
17. B
18. C
19. A
20. C
21. E
22. C
23. B
24. A
25. A
26. B
27. C
28. B
29. D
30. D

Solutions:

$$1. \log_2 128 - \log_3 27 = 7 - 3 = 4$$

$$2. 5^x = 0.04 = \frac{1}{25} = 5^{-2} \Rightarrow x = -2$$

$$3. \log_2 \left(2048^{\frac{1}{5}} \right) = \log_2 \left(\left(2^{11} \right)^{\frac{1}{5}} \right) = \log_2 \left(2^{\frac{11}{5}} \right) = \frac{11}{5}$$

$$4. 2500 = 5^4 \cdot 2^2, \text{ so the greatest power of 5 that divides 2500 is } 5^4; \text{ hence, } x = 4.$$

$$5. \log_9 2916 = \log_{3^2} 54^2 = \log_3 54 = \log_3 (3^3 \cdot 2) = 3 \log_3 3 + \log_3 2 = 3 + a$$

$$6. 3^{2x-1} = \frac{1}{9} = 3^{-2} \Rightarrow 2x-1 = -2 \Rightarrow x = -\frac{1}{2}$$

$$7. 2^{3 \log_2 3} = 2^{\log_2 3^3} = 2^{\log_2 27} = 27$$

$$8. 2^{2x} = (2^2)^x = 4^x = 8^{x-1} = (2^3)^{x-1} = 2^{3x-3} \Rightarrow 2x = 3x-3 \Rightarrow x = 3$$

$$9. \log_3 \left(\frac{81^{x+1}}{9^{2x-3}} \right) = \log_3 \left(\frac{(3^4)^{x+1}}{(3^2)^{2x-3}} \right) = \log_3 \left(\frac{3^{4x+4}}{3^{4x-6}} \right) = \log_3 (3^{(4x+4)-(4x-6)}) = \log_3 (3^{10}) = 10$$

10. Since the inequality is satisfied by every negative integer, there are an infinite number of integer solutions to the equation.

11. $1 < \ln 3 < 2 \Rightarrow 0 < \ln(\ln 3) < \ln 2 < 1 \Rightarrow \ln(\ln(\ln 3)) < 0 \Rightarrow \ln(\ln(\ln(\ln 3)))$ is undefined, so the maximum number of times f can be composed and still be defined on 3 is 3 times.

$$12. \log_b 81 = 4 \Rightarrow b^4 = 81 \Rightarrow b = 3 \Rightarrow \log_3 b = \log_3 3 = 1$$

$$13. \sum_{k=2}^{10} 2^k = \frac{4(1-2^9)}{1-2} = \frac{4(-511)}{-1} = 2044$$

14. $1 = \log_8(x-3) + \log_8(x-5) = \log_8(x-3)(x-5) = \log_8(x^2 - 8x + 15) \Rightarrow x^2 - 8x + 15 = 8$
 $\Rightarrow 0 = x^2 - 8x + 7 = (x-7)(x-1) \Rightarrow x = 7 \text{ or } x = 1$, but 1 makes the arguments of the logarithms negative while 7 does not, so the only solution to the equation, and hence the sum of the solutions to the equation, is 7.

15. $8 \log_x 2 = \log_x(2^8) = \log_x 256 = \log_2 x \Rightarrow 8 = (\log_2 x)^2 \Rightarrow \log_2 x = \pm 2\sqrt{2} \Rightarrow x = 2^{\pm 2\sqrt{2}}$, and both values are positive and not equal to 1, so both are solutions. The product, therefore, is $(2^{2\sqrt{2}})(2^{-2\sqrt{2}}) = 2^{2\sqrt{2}-2\sqrt{2}} = 2^0 = 1$.

16. The greatest power of 3 less than 2026 is $3^6 = 729$, and the least power of 3 greater than 2026 is $3^7 = 2187$. Therefore, $m + n = 6 + 7 = 13$.

17. $150 = 5^x + 5^{x-1} = 5^x + \frac{5^x}{5} = 5^x \left(1 + \frac{1}{5}\right) = 5^x \left(\frac{6}{5}\right) \Rightarrow 5^x = 125 \Rightarrow x = 3$

18. The number of factors of 3 in 100! is $\left\lfloor \frac{100}{3} \right\rfloor + \left\lfloor \frac{100}{9} \right\rfloor + \left\lfloor \frac{100}{27} \right\rfloor + \left\lfloor \frac{100}{81} \right\rfloor = 33 + 11 + 3 + 1 = 48$.

19. $648 = 3^x - 3^{x-2} = 3^x - \frac{3^x}{3^2} = 3^x \left(1 - \frac{1}{9}\right) = 3^x \left(\frac{8}{9}\right) \Rightarrow 3^x = 729 \Rightarrow x = 6$

20. Write $a = \log_2 x$, then $\frac{1}{a} = \log_x 2$, and $(\log_2 x)^2 + (\log_x 2)^2 = a^2 + \frac{1}{a^2} = \left(a + \frac{1}{a}\right)^2 - 2$
 $= 7^2 - 2 = 47$.

21. $x^{11} = x^{6+3+2} = x^6 x^3 x^2 = x^{\log_2 64} x^{\log_4 64} x^{\log_8 64} = 64^{\log_2 x} 64^{\log_4 x} 64^{\log_8 x} = 64^{\log_2 x + \log_4 x + \log_8 x} = 64^n$
 $\Rightarrow x = 64^{\frac{n}{11}}$, and since 11 is prime and 64 is not a perfect 11th power, the only time this would be an integer is if n is a multiple of 11. Therefore, 22 is the next value of n for which this equation has an integer solution.

22. Since $12 = 2^2 \cdot 3$, to find the number of consecutive zeros at the end of $2026!$, we must find how many factors of 2 and 3 are in $2026!$. The number of factors of 2 is $\left\lfloor \frac{2026}{2} \right\rfloor + \left\lfloor \frac{2026}{4} \right\rfloor$

$$+ \left\lfloor \frac{2026}{8} \right\rfloor + \left\lfloor \frac{2026}{16} \right\rfloor + \left\lfloor \frac{2026}{32} \right\rfloor + \left\lfloor \frac{2026}{64} \right\rfloor + \left\lfloor \frac{2026}{128} \right\rfloor + \left\lfloor \frac{2026}{256} \right\rfloor + \left\lfloor \frac{2026}{512} \right\rfloor + \left\lfloor \frac{2026}{1024} \right\rfloor = 1013 + 506$$

$+253 + 126 + 63 + 31 + 15 + 7 + 3 + 1 = 2018$. The number of factor of 3 is $\left\lfloor \frac{2026}{3} \right\rfloor + \left\lfloor \frac{2026}{9} \right\rfloor$

$\left\lfloor \frac{2026}{27} \right\rfloor + \left\lfloor \frac{2026}{81} \right\rfloor + \left\lfloor \frac{2026}{243} \right\rfloor + \left\lfloor \frac{2026}{729} \right\rfloor = 675 + 225 + 75 + 25 + 8 + 2 = 1010$. Each factor of 12

requires two 2s and one 3, so pairing those up, we would run out of 2s just before we ran out of

3s. Therefore, the number of factors of 12 in $2026!$ is $\frac{2018}{2} = 1009$.

23. $0 = 9^x - 4 \cdot 3^x + 3 = (3^x)^2 - 4 \cdot 3^x + 3 = (3^x - 1)(3^x - 3) \Rightarrow x = 0$ or $x = 1$, so the sum of the solutions is 1.

24. $1024 = 2^{10} \leq n^2 + n + 1 < 2^{11} = 2048 \Rightarrow 1023 \leq n(n+1) < 2047$. Since the quantity in the middle of the inequality increase for increasing positive n , the least positive n that works is 32 and the greatest positive n that works is 44, for a total of 13 integers. For increasing negative n , the least negative n that works is -45 and the greatest negative n that works is -33 , for a total of 13 more integers. Therefore, 26 total integers fit this inequality.

25. $\prod_{k=2}^{2047} (\log_{k^2} (k+1))^2 = \prod_{k=2}^{2047} (\log_k (k+1)) = \log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdots \log_{2047} 2048 = \log_2 2048 = 11$

26. These numbers are small enough that we can check the powers of 5 until we get a power of 2 that works. Starting with $n=2$, $2^m = 5^2 + 3 = 25 + 3 = 28$, which is not a power of 2. If $n=3$, $2^m = 5^3 + 3 = 125 + 3 = 128 \Rightarrow m = 7$, so the least possible sum is $7 + 3 = 10$.

27. 7^4 ends in 01, so $7^{45^2+1} = 7^{2026} = 7^{4 \cdot 506+2} = (7^{4 \cdot 506})(7^2) = (7^4)^{506}(49)$, so this ends in $(01)^{506}(49) = 49$, making the tens digit 4.

28. $\frac{\log_{10} 75 - \log_{100} 9}{\log_{1000} 125} = \frac{\log_{10} 75 - \log_{10^2} (3^2)}{\log_{10^3} (5^3)} = \frac{\log_{10} 75 - \log_{10} 3}{\log_{10} 5} = \frac{\log_{10} \frac{75}{3}}{\log_{10} 5} = \frac{\log_{10} 25}{\log_{10} 5} = \log_5 25 = 2$

29. $3^{20} = 10^{\log(3^{20})} = 10^{20 \log 3} \approx 10^{20(0.48)} = 10^{9.6} = 10^{0.6+9} = 10^{0.6} 10^9 \Rightarrow 1 \cdot 10^9 < 10^{0.6} 10^9 < 10 \cdot 10^9$, and since $1 \cdot 10^9$ has 10 digits, while $10 \cdot 10^9$ is the least integer with 11 digits, this number has 10 digits (for the record, $3^{20} = 3,486,784,401$, which has 10 digits).

30. All of the exponentials in this equation have bases where 2 and 5 are the only factors, so this presumably factors using exponentials with bases of 2 and 5. Rewrite the equation in that

$$\text{manner: } 0 = 2^x (5^x)^2 - 3 \cdot (5^x)^2 - 2 \cdot 2^x \cdot 5^x + 6 \cdot 5^x - 8 \cdot 2^x + 24 = (5^x)^2 (2^x - 3) - 2 \cdot 5^x (2^x - 3)$$

$$-8(2^x - 3) = (2^x - 3)((5^x)^2 - 2 \cdot 5^x - 8) = (2^x - 3)(5^x - 4)(5^x + 2), \text{ so the only real solutions to the}$$

equation are $\log_2 3$ and $\log_5 4$, making the product of the solutions $\log_2 3 \cdot \log_5 4 = \log_2 4 \cdot \log_5 3$

$$= 2 \log_5 3 = \log_5 (3^2) = \log_5 9. \text{ 5 is already prime, so it is not a power of a smaller positive}$$

integer, making this expression the one with the least possible base and argument, giving it the least possible sum of $5 + 9 = 14$.