

- 1) D
- 2) E
- 3) D
- 4) E
- 5) B

- 6) C
- 7) A
- 8) C
- 9) C
- 10) D

- 11) D
- 12) E
- 13) A
- 14) D
- 15) C

- 16) C
- 17) E
- 18) C
- 19) E
- 20) A

- 21) C
- 22) B
- 23) C
- 24) B
- 25) D

- 26) B
- 27) B
- 28) B
- 29) A
- 30) C

1) The prices of the products are $65 \cdot \frac{4}{10}$, $55 \cdot \frac{5}{10}$, $45 \cdot \frac{6}{10}$, and $35 \cdot \frac{7}{10}$. Eliminating $5 \cdot \frac{1}{10}$ from all of these gives $13 \cdot 4 = 52$, $11 \cdot 5 = 55$, $9 \cdot 6 = 54$, and $7 \cdot 7 = 49$. The cheapest product is the Rubik's cube. **[D]**

2) If a root is given by the Rational Root Theorem, then so is its negative, so all potential roots will always add to 0. **[E]**

3) Any positive rational root $\frac{f_n}{f_0}$ will be of the form $2^a \cdot 3^b \cdot 5^c$ where the exponents can be increased by the value in the numerator and decreased by the value in the denominator. Thus, a ranges from -3 to 1 , b ranges from -1 to 1 , and c ranges from 0 to 2 . The number of possible positive rational roots is therefore $5 \cdot 3 \cdot 3 = 45$. There are an equal number of negative rational roots, so the total number of possible rational roots is 90. **[D]**

4) Each element in the list of all possible rational roots must contain $5^0, 5^1$, or 5^2 ; $2^{-3}, 2^{-2}, 2^{-1}, 2^0$, or 2^1 ; and $3^{-1}, 3^0$, or 3^1 . This makes $45 = 3 \cdot 5 \cdot 3$ positive factors, along with 45 negative factors for a total of 90 factors. Thirty of those factors contain each of $3^{-1}, 3^0$, and 3^1 , so those cancel, meaning $y = 0$. By similar reasoning, the power of 5 in the product will be $(5^0)^{30} \cdot (5^1)^{30} \cdot (5^2)^{30} = 5^{90}$, so $z = 90$. Again, by similar reasoning, since there are 18 factors containing each of the powers of 2, the power of 2 in the product will be $(2^{-3})^{18} \cdot (2^{-2})^{18} \cdot (2^{-1})^{18} \cdot (2^0)^{18} \cdot (2^1)^{18} = 2^{-90}$, so $x = -90$. Lastly, because the number of negative rational roots is odd, the overall product has negative sign, so $n = 1$. Therefore, $1(-90 + 0 + 3 \cdot 90) = 180$. **[E]**

5) Let the volume of Ursula's mocha be x , and let the volume of Croix's mocha be y . Then $\frac{4x}{5} + \frac{y}{2} = 8$ and $\frac{x}{5} + \frac{y}{2} = 4$. Subtracting these, $\frac{3x}{5} = 4$ and $x = \frac{20}{3}$, so $y = \frac{16}{3}$. $16 + 3 = 19$. **[B]**

6) $\tan(\theta + \varphi) = \frac{\tan \theta + \tan \varphi}{1 - \tan \theta \cdot \tan \varphi} = \frac{6}{1 - \tan \theta \cdot \tan \varphi}$. This value is minimized when $\tan \theta \cdot \tan \varphi$ is maximized. By AM-GM, $\frac{\tan \theta + \tan \varphi}{2} = 3 \geq \sqrt{\tan \theta \cdot \tan \varphi}$, so $\tan \theta \cdot \tan \varphi \leq 9$ and this is achieved when $\theta = \varphi = \arctan 3$. $\frac{1}{1 - 3 \cdot 3} = -\frac{3}{4}$. **[C]**

7) The lines form angles of $\theta_1 = \arctan 1$ and $\theta_2 = \arctan 7$ with the x -axis. We wish to find $\tan\left(\frac{\theta_1 + \theta_2}{2}\right)$. $\tan(\theta_1 + \theta_2) = \frac{1+7}{1-1 \cdot 7} = -\frac{4}{3}$. This is a second-quadrant angle, so $\cos(\theta_1 + \theta_2) = -\frac{3}{5}$. $\tan\left(\frac{\theta_1 + \theta_2}{2}\right) = \sqrt{\frac{1+\cos(\theta_1 + \theta_2)}{1-\cos(\theta_1 + \theta_2)}} = \sqrt{\frac{1-3/5}{1+3/5}} = \sqrt{\frac{2}{5}} = \frac{\sqrt{10}}{5}$. **[A]**

8) $4,101,649 = 2025^2 + 1024$. Using the Sophie-Germain factorization, this is equivalent to $45^4 + 4 \cdot 4^4 = (45^2 + 2 \cdot 45 \cdot 4 + 2 \cdot 4^2)(45^2 + 2 \cdot 45 \cdot 4 + 2 \cdot 4^2) = 2417 \cdot 1697$. The sum of these is 4114. 4110 is a multiple of 6, so $4114 \bmod 6 = 4$. **[C]**

9) The angle of rotation of the ellipse can be found by solving $\cot(2\theta) = \frac{8-6}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$, so $\theta = \frac{\pi}{6}$. The minor axis is on the line $r = \frac{\pi}{6}$ ($y = \frac{x}{\sqrt{3}}$) or the line perpendicular to this, $y = -\sqrt{3}x$. Investigating

the first line, $8x^2 + 2x^2 + 2x^2 = 9$, so $x = \frac{\sqrt{3}}{2}$ and $y = \frac{1}{2}$ for an axis length of 2. Investigating the second line, $8x^2 - 6x^2 + 18x^2 = 9$, so $x = \frac{3\sqrt{5}}{10}$ and $y = -\frac{3\sqrt{15}}{10}$ for an axis length of $\frac{6}{\sqrt{5}}$. The shorter axis has length 2. [C]

10) Let $A = (0,0)$, $H = (x, -1)$, and $S = (\sqrt{3}, y)$. S is a 60° clockwise rotation of H about A .

Using complex numbers, this means $S = \sqrt{3} + yi = (x - i)\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \left(\frac{x}{2} - \frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}x\right)i$, so $\frac{x}{2} - \frac{\sqrt{3}}{2} = \sqrt{3}$ and $x = 3\sqrt{3}$ and $y = -5$. AHS has side length $2\sqrt{7}$ and area $7\sqrt{3}$.

[D] Note: An alternate solution involving three applications of the Pythagorean Theorem exists, leading to finding the 5 as the root of a quartic polynomial.

11) a_n , b_n , and c_n are the roots of the polynomial $x^3 - (3n - 1)x^2 + (3n + 1)x - 3 = 0$. By inspection, 1 is a root of this polynomial, which factors as $(x - 1)(x^2 + (2 - 3n)x + 3)$. The other roots of the polynomial are $\frac{3n - 2 \pm \sqrt{9n^2 - 12n - 8}}{2}$. Since $\sqrt{9n^2 - 12n - 8} \approx 3n - 2$, the largest root is approximately $3n - 2$ as n grows large. $\lim_{n \rightarrow \infty} \frac{3n - 2}{n} = 3$. [D]

12) The sum of the measures of the exterior angles of a triangle is 360° , so solving $3x + 4x + 5x = 360$ gives the measures of the exterior angles as 90° , 120° , and 150° . Thus, the smallest interior angle has measure 30° . [E]

13) Squaring, $A + B + 2\sqrt{AB} = 14 + 8\sqrt{3} = 14 + 2\sqrt{48}$. A and B are 6 and 8 in some order, so $|A - B| = 2$. [A]

14) Let the first term of the series be a , and let the common ratio be r . Then $\frac{a}{1-r} = 20$ and $\frac{a}{1-2r} = 26$. This creates the system $a + 20r = 20$, $a + 52r = 26$. Subtracting, $32r = 6$ so $r = \frac{3}{16}$ and $a = \frac{65}{4}$. $65 + 4 = 69$. [D]

15). The sum of the squares of the roots is 1. By Vieta's, this is the square of the sum of the roots minus twice the product of the roots. Solving $\left(\frac{13}{10}\right)^2 - \frac{2c}{20} = 1$ gives $c = \frac{69}{10}$. [C]

16) Let $g(x) = f(x) - 25$. Then $g(x)$ has roots at 5 and 9, so by the Integer Root Theorem the constant term of $g(x)$ must be a multiple of 45. Thus, the constant term of $f(x)$ is of the form $45n + 25$ for some integer n . The smallest magnitude of this is at $n = -1$, which is 20. [C]

17) Clearly, $m + n = \sum_{n=1}^{\infty} \frac{1}{n^5}$; call this S . $\frac{S}{32} = \sum_{n=1}^{\infty} \frac{1}{32n^5} = \sum_{n=1}^{\infty} \frac{1}{(2n)^5} = n$. Then $m = \frac{31S}{32}$ and $\frac{m}{n} = 31$. The remainder when this is divided by 6 is 1. [E]

18) Consider a cross-section of the figure. This is a semicircle (call its center O) inscribed in an isosceles triangle (ABC with the apex at A) with height 7. Drawing a radius of the circle to the leg

(intersecting at D), similar triangles are created. $AOC \sim ADO$, so $\frac{AC}{AO} = \frac{AO}{AD}$. By the Pythagorean Theorem, $AD = \sqrt{7^2 - 1^2} = 4\sqrt{3}$, so solving $\frac{AC}{7} = \frac{7}{4\sqrt{3}}$ gives $AC = \frac{49\sqrt{3}}{12}$. [C]

19) $a^2 + 2026^2 = c^2$ for positive integers a and c . Then $c^2 - a^2 = 2026^2$. By Difference of Squares, $(c + a)(c - a) = 2^2 \cdot 1013^2$. $c + a$ and $c - a$ must share the same parity, which is only possible if they are both even and $c + a = 2 \cdot 1013^2$ and $c - a = 2$. Solving this system gives $c = 1013^2 + 1$ and $a = 1013^2 - 1$. Note that a is even. Additionally, 1011 is a multiple of 3, so a is congruent to $2^2 - 1$ modulo 3, so a is divisible by 6 and leaves a remainder of 0. [E]

20) If the sides of the triangle were x , x , and $\frac{5x}{2}$, then this would fail the Triangle Inequality. Thus, the sides are x , $\frac{5x}{2}$, and $\frac{5x}{2}$. The perimeter is $6x = 36$, so $x = 6$ and the sides are 6, 15, and 15. $6 + 15 = 21$. [A]

21) Taking the logarithm base 2 of both sides, $(3 - \log_2 x) \log_2 x = \log_2 \pi$. Let $y = \log_2 x$ and $k = \log_2 \pi$. Then $(3 - y)y = k$ and $y^2 - 3y + k = 0$. Since $k < \frac{9}{4}$, this has real solutions. The product of the possible values of x is 2 to the power of the sum of the possible values of y , which is 3 by Vieta's. $2^3 = 8$. [C]

22) Note that the desired value is $f(10)$. The Method of Finite Differences can be used.

20	26	2026	6090
6	2000	3994	
	1994	1994	

$f(10) = 6020$. [B]

23) When $x \geq \frac{1}{a}$, the graphs are $y = ax - 1$ and $y = bx$ and intersect at the point $(\frac{1}{a-b}, \frac{b}{a-b})$. When $x < \frac{1}{a}$, the graphs are $y = 1 - ax$ and $y = bx$ and intersect at the point $(\frac{1}{a+b}, \frac{b}{a+b})$. Either these must be the same point, or one of the points does not exist; these are respectively resolved when $b = 0$ (which is impossible) or $a = b$. Thus, the second point is the one where $x < \frac{1}{a}$ and is the point $(\frac{1}{2a}, \frac{1}{2})$. Only the x -coordinate depends on a . [C]

24) Let $x = \frac{1}{1+\frac{1}{1+\dots}}$. Through recursion, $x = \frac{1}{1+x}$, so $x^2 + x - 1 = 0$ and $x = \frac{-1+\sqrt{5}}{2}$. Now let $y = \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}$. Similarly, $y = \sqrt{1 + y}$, so $y^2 - y - 1 = 0$ and $y = \frac{1+\sqrt{5}}{2}$. $x - y = -1$. [B]

25) BD is the third side of one triangle that contains BC and CD and another that contains AB and DA . These are triangles with side lengths 5 and 7, and 3 and 13 respectively. By the Triangle Inequality, $2 < BD < 12$ and $10 < BD < 16$. The only possible integer value of BD is 11. [D]

26) All letters other than the five mentioned can be ignored. The possible orientations of the relevant letters are 'COUNT' or 'COUTN'. There are $5! = 120$ ways to arrange the letters in total, so the probability is $\frac{1}{60}$. **[B]**

$$27) AB = \begin{bmatrix} 1+0+2 \\ 2+0+2 \\ 2+0+3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}. 3+4+5 = 12. \text{ [B]}$$

$$28) \text{ Investigating } A(AB) \text{ gives } \begin{bmatrix} 3-8+10 \\ 6-4+10 \\ 6-8+15 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \\ 13 \end{bmatrix}, \text{ another Pythagorean triple. For further}$$

confirmation, left-multiplying this by A yields $\begin{bmatrix} 7 \\ 24 \\ 25 \end{bmatrix}$ and $(a-2b+2c)^2 + (2a-b+2c)^2 =$

$$(2a-2b+3c)^2 + (a^2+b^2-c^2), \text{ so if } P = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ is a Pythagorean triple then so is } AP. A^5B \text{ will}$$

give the Pythagorean triple with hypotenuse 1 greater than the longest leg with shorter leg $1+5 \cdot 2 = 11$. The relevant triple is $11-60-61$, so the largest entry of A^5B is 61. **[B]**

$$29) \frac{1}{38} \text{ of the time, the casino loses } \$35. \frac{37}{38} \text{ of the time, the casino wins } \$1. \frac{-35}{38} + \frac{37}{38} = \frac{2}{38}. \text{ [A]}$$

30) Despite the probability of winning being greater than the probability of losing, the amount lost on a loss is sufficient to offset the lower probability. If p is the probability of winning, then $E = pW + (1-p)L$; note the scaling of $1-p$ (the probability of losing) by the amount lost. **[C]**