

Answer Key:

1. C
2. D
3. A
4. D
5. E

6. A
7. C
8. B
9. A
10. A

11. B
12. D
13. C
14. E
15. B

16. B
17. D
18. A
19. B
20. E

21. B
22. A
23. A
24. B
25. A

26. D
27. C
28. B
29. D
30. E

Solutions:

1. **C:** By row operations, $\begin{vmatrix} 2026 & 2024 \\ 2025 & 2023 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2025 & 2023 \end{vmatrix} = 2023 - 2025 = -2$.
2. **D:** By subtracting the two matrices and dividing the entries by 2, we get that $B = \begin{bmatrix} 2 & -6 \\ -1 & 4 \end{bmatrix}$ with a determinant of 2, giving us an inverse of $\frac{1}{2} \begin{bmatrix} 4 & 6 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ \frac{1}{2} & 1 \end{bmatrix}$.
3. **A:** To multiply matrices, you multiply the m^{th} row of the first by the n^{th} column of the second to get the m^{th} row, n^{th} column entry of the product, resulting in $\begin{bmatrix} 1 & 15 \\ -13 & 3 \end{bmatrix}$.
4. **D:** Two vectors are perpendicular when their dot product is 0. Taking the dot product with the given vector and each of the answer choices, we get 2, -4, -30, and 0, meaning $\langle 2, 4, 7 \rangle$ is perpendicular.
5. **E:** Note that the given matrix is the clockwise rotation matrix for $\theta = \frac{7\pi}{6}$. As such, A should be the clockwise rotation matrix for $\theta = \frac{7\pi}{6} \times 2026 = \frac{7091\pi}{3}$, which is coterminal to $\frac{5\pi}{3}$. So, $A_{1,2} = -\sin\left(\frac{5\pi}{3}\right) = \frac{\sqrt{3}}{2}$.
6. **A:** The cosine of the angle between two vectors is given by $\cos \theta = \frac{u \cdot v}{||u|| ||v||}$, which is $\frac{24}{5\sqrt{2} \times 4\sqrt{2}} = \frac{3}{5}$.
7. **C:** The projection of a vector is given by: $\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2} * \vec{v}$, $\frac{10}{(\sqrt{35})^2} * \langle -3, 5, 1 \rangle = \frac{1}{7} \langle -6, 10, 2 \rangle$.
8. **B:** $\begin{bmatrix} 1 & 0 & -2 \\ 3 & -1 & 0 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 3 & -1 & 0 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & -2 & -6 \\ 0 & 1 & -6 \\ 9 & 1 & 0 \end{bmatrix}$. The sum of the elements is -6.
9. **A:** The product of the eigenvalues is equivalent to the determinant of the matrix. $|M| = -12$
10. **A:** The inverse of a 3×3 matrix is the transpose of its cofactor matrix multiplied by the reciprocal of the determinant of the original matrix. The cofactor matrix of M is $\begin{bmatrix} -2 & -6 & 5 \\ -2 & 6 & -1 \\ -2 & -6 & -1 \end{bmatrix}$, making the inverse $P = \frac{1}{-12} \begin{bmatrix} -2 & -2 & -2 \\ -6 & 6 & -6 \\ 5 & -1 & -1 \end{bmatrix}$ with $P_{1,3} = \frac{1}{6}$
11. **B:** Taking vectors from the first vertex to the other vertices yields $\langle 1, 3, 7 \rangle$, $\langle 3, -2, -5 \rangle$, and $\langle -2, 9, 4 \rangle$. The volume of the tetrahedron is given by $\left| \frac{1}{6} \begin{vmatrix} 1 & 3 & 7 \\ 3 & -2 & -5 \\ -2 & 9 & 4 \end{vmatrix} \right| = 32$
12. **D:** For a system of equations to have no solution, its coefficient matrix must have a determinant of 0. By solving $\begin{vmatrix} A & 1 & -2 \\ 3 & A & 3 \\ -8 & -2 & A \end{vmatrix} = A^3 - 13A - 12 = 0$, we get solutions of -3, -1, and 4. Note, however, that $A = 4$ makes the first and third equation the same, meaning there would be infinitely many solutions instead. Thus, our answer is $-3 - 1 = -4$.

13. C: For an $n \times n$ matrix A , $|3A| = 3^n|A|$, so A must be a 3×3 matrix. Given this, $|-2A^{-1}| = (-2)^3|A^{-1}| = -\frac{8}{6} = -\frac{4}{3}$

14. E: A) The determinant of A is zero, meaning it has no inverse
 B) AB is a 2×3 matrix, while CA is a 3×2 matrix, matrices of different sizes cannot be added
 C) $B^T A$ results in a 3×2 matrix, which cannot be multiplied by C , another 3×2 matrix
 D) $B^T C^T$ results in a 3×3 matrix, which cannot be added to A , a 2×2 matrix
 As none of the answer choices are defined, the answer must be E

15. B: By picking any point on the first plane (such as $(0, 0, 2)$) and finding its distance to the plane $4x - y + 8z - 7 = 0$, we get $\frac{9}{\sqrt{81}} = 1$

16. B: The determinant on the left can be rearranged to the determinant on the right by the following operations: $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \rightarrow \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} \rightarrow \begin{vmatrix} a+2g & d & g \\ b+2h & e & h \\ c+2i & f & i \end{vmatrix} \rightarrow \begin{vmatrix} a+2g & d & g \\ c+2i & f & i \\ b+2h & e & h \end{vmatrix} \rightarrow \begin{vmatrix} 3a+6g & 3d & 3g \\ c+2i & f & i \\ b+2h & e & h \end{vmatrix}$. The operations are a transpose, adding a scalar multiple of one row to another, switching two rows, and scaling a row by 3. The first two operations do not change the determinant, switching rows changes the sign of the determinant, and scaling a row by 3 multiplies the determinant by 3. Thus, our answer is $4 \times -3 = -12$

17. D: An $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors.

- $L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \rightarrow \lambda^2 - 1 = 0 \rightarrow \lambda = \pm 1$. For $\lambda = 1$, $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_2 = x_1 \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector. For $\lambda = -1$, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_2 = -x_1 \rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector. Thus, L has 2 linearly independent eigenvectors and hence, is diagonalizable.
- M is a diagonal matrix, so it is diagonalizable.
- $N = \begin{bmatrix} 6 & -1 \\ 1 & 4 \end{bmatrix} \rightarrow \begin{vmatrix} 6-\lambda & -1 \\ 1 & 4-\lambda \end{vmatrix} = (6-\lambda)(4-\lambda) - (-1) = 0 \rightarrow \lambda^2 - 10\lambda + 25 = 0 \rightarrow (\lambda - 5)^2 = 0 \rightarrow \lambda = 5$. For $\lambda = 5$, $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_2 = x_1 \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector. Since $\lambda = 5$ is the only distinct eigenvalue and only yields one linearly independent eigenvector, N is not diagonalizable.
- $O = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0 \rightarrow (\lambda - 1)^2 = 0 \rightarrow \lambda = 1$. For $\lambda = 1$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_2 = 0 \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector. Since $\lambda = 1$ is the only distinct eigenvalue and only yields one linearly independent eigenvector, O is not diagonalizable.

18. A: The column vectors of A are the image of the standard basis vectors after the reflection. The image of $\vec{i}$ is found by reflecting the point $(1, 0)$ across the line $y = 2x$. The perpendicular line containing $(1, 0)$ is $y = -\frac{1}{2}x + \frac{1}{2}$ and substituting yields the intersection point $(\frac{1}{5}, \frac{2}{5})$. Since this point is the midpoint of $(1, 0)$ and its image, the change in coordinates must be the same. Thus, the image is the point $(-\frac{3}{5}, \frac{4}{5})$. By the same method, using the perpendicular line $y = -\frac{1}{2}x + 1$ and intersection point $(\frac{2}{5}, \frac{4}{5})$, the image of $(0, 1)$ is

$(\frac{4}{5}, \frac{3}{5})$. This gives us the image of $\vec{j}$. Thus, $A = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$, and hence, $\frac{50abc}{d} = \frac{50}{1} \times \frac{-3}{5} \times \frac{4}{5} \times \frac{4}{5} \times \frac{5}{3} = -32$.

19. B: For symmetric $n \times n$ matrixes A and B , the product AB is symmetric if and only if $AB = BA$. ($\Rightarrow$) If AB is symmetric, then $AB = (AB)^T = B^T A^T = BA$. ($\Leftarrow$) If $AB = BA$, then $AB = BA = B^T A^T = (AB)^T$. The other three options do not provide enough relevant information about A and B to show both implications.

20. E: The nested inverses translate to the equation $A = 2I + A^{-1}$. This is equivalent to $A^2 = 2A + I \rightarrow A^2 - 2A - I = 0$. For A to be a solution to this equation with nonnegative entries, it must be a diagonal matrix whose nonzero elements are the positive root of the polynomial $x^2 - 2x - 1 = 0$. This root is

$x = 1 + \sqrt{2}$. Thus, $A = \begin{bmatrix} 1 + \sqrt{2} & 0 \\ 0 & 1 + \sqrt{2} \end{bmatrix}$ and hence, $\det(A) = (1 + \sqrt{2})^2 = 3 + \sqrt{2}$.

21. B: Using elementary row operations, $\begin{bmatrix} 2 & 3 & -5 & 10 \\ -1 & -1 & 1 & -3 \\ 1 & 2 & -4 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -4 & 7 \\ -1 & -1 & 1 & -3 \\ 2 & 3 & -5 & 10 \end{bmatrix} \rightarrow$
 $\begin{bmatrix} 1 & 2 & -4 & 7 \\ 0 & 1 & -3 & 4 \\ 0 & -1 & 3 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -4 & 7 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

22. A: The null space is the set of vectors that are collapsed to the origin by the transformation represented by A . Using the reduced row echelon form, it is the set of solutions to

$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. So, $x_1 + 2x_3 - x_4 = 0$, $x_2 - 3x_3 + 4x_4 = 0 \rightarrow x_1 = -2x_3 + x_4$,

$x_2 = 3x_3 - 4x_4$ where x_3 and x_4 are free variables. So, the solutions are given by $\begin{bmatrix} -2x_3 + x_4 \\ 3x_3 - 4x_4 \\ x_3 \\ x_4 \end{bmatrix}$. For

$x_3 = 1, x_4 = 0$, this is $\begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ and for $x_3 = 0, x_4 = 1$, this is $\begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}$. Thus, $\left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the null space of A .

23. A: The normal vector to the plane is $\vec{n} = \langle 1, 2, -2 \rangle$. Then, by properties of vector addition, we can see that the reflection of $\vec{v}$ across the plane is given by $\vec{v} - 2\text{proj}_{\vec{n}}\vec{v}$. $\text{proj}_{\vec{n}}\vec{v} = \frac{-6}{3^2} \langle 1, 2, -2 \rangle = \langle \frac{-2}{3}, \frac{-4}{3}, \frac{4}{3} \rangle$. So, $\vec{v} - 2\text{proj}_{\vec{n}}\vec{v} = \langle 2, -1, 3 \rangle - 2 \langle \frac{-2}{3}, \frac{-4}{3}, \frac{4}{3} \rangle = \langle \frac{10}{3}, \frac{5}{3}, \frac{1}{3} \rangle$.

24. B: The only distinct eigenvalue of a nilpotent matrix is 0. Since the trace is equal to the sum of the eigenvalues, the trace must be 0. This property also eliminates the possibility of A and D, since the determinant is the product of the eigenvalues. Not all nilpotent matrices are diagonalizable such as $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, eliminating C.

25. A: $\begin{vmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 4 & 0 & 0 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & -2 \\ 4 & 0 & 0 & 1 \end{vmatrix}$. Since the first 4×4 matrix is

upper triangular, its determinant is the product of its main diagonal elements, which is 1. The determinant of the second 4×4 is equal to $-(-1) \begin{vmatrix} 0 & 3 & 0 \\ 0 & 1 & -2 \\ 4 & 0 & 1 \end{vmatrix} = -24$. Thus our final answer is $1 - 2(-24) = 49$.

26. D: First we must find the vector equation of the line. The direction vector is given by the cross product

of the two perpendicular vectors, which is $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 4 & -1 & 2 \end{vmatrix} = 3\vec{i} - 6\vec{j} - 9\vec{k}$. Dividing by 3 to make things

easier, we get the equation $\vec{r}(t) = \langle 1, -2, -3 \rangle t + \langle 2, -1, 3 \rangle = \langle t + 2, -2t - 1, -3t + 3 \rangle$. Plugging these components into the plane equation yields $t + 2 - 2t - 1 - 3t + 3 = 6 \rightarrow t = -\frac{1}{2}$. Plugging this value back into the line equation yields the intersection point $(\frac{3}{2}, 0, \frac{9}{2})$.

27. C: The transition matrix is $\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$. To solve for the state vector when the system is stable, we must solve for a and b such that $\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $a + b = 1$. The product yields dependent equations $\begin{cases} 0.8a + 0.3b = a \\ 0.2a + 0.7b = b \end{cases} \rightarrow 0.2a - 0.3b = 0$. Eliminating with $0.3a + 0.3b = 0.3$ yields $0.5a = 0.3 \rightarrow a = 0.6$

28. B: The trace (sum of main diagonal elements) is given by $4 + 8 + 16 + \dots + 80 = 10(4 + 80) = 840$

29. D: Since $\vec{a}$ and $\vec{b}$ are perpendicular to $\vec{a} \times \vec{b}$, $(\vec{a} \times \vec{b}) \cdot \vec{a} = \vec{0}$ and $(\vec{a} \times \vec{b}) \cdot \vec{b} = \vec{0}$. Thus, $(\vec{a} \times \vec{b}) \cdot (\vec{a} + \vec{b}) = \vec{0}$. Option A is not necessarily true because we cannot determine anything about the relationship between $\vec{c}$ and $\vec{a} \times \vec{b}$. Option B is not necessarily true because the cross product is anticommutative. Since the cross product of two vectors is equal to $\vec{0}$ only if they are linearly dependent, Option C would only be true if $\vec{a} = \vec{0}$. So, it is not necessarily true.

30. E: By Cramer's Rule, $x = \frac{\begin{vmatrix} \cot(\theta) & \cos(\theta) \\ \tan(\theta) & -\sin(\theta) \end{vmatrix}}{\begin{vmatrix} \sin(\theta) & \cos(\theta) \\ \cos(\theta) & -\sin(\theta) \end{vmatrix}} = \frac{-\cot(\theta)\sin(\theta) - \tan(\theta)\cos(\theta)}{-(\sin^2(\theta) + \cos^2(\theta))} = \frac{-\cos(\theta) - \sin(\theta)}{-1} = \cos(\theta) + \sin(\theta)$.

Since the determinant of the coefficient matrix is nonzero independent of θ , we know that this is part of a unique solution.