

Answers:

<u>Relay 1</u> 1. -5 2. 169 3. 42 4. 51 5. 72 6. -17	<u>Relay 4</u> 1. 10,404 2. 1156 3. 23 4. 252 5. 11 6. 1
<u>Relay 2</u> 1. 12 2. 86 3. 274 4. 25 5. 10 6. 4	<u>Relay 5</u> 1. 0 2. 9 3. -35 4. 10 5. 924 6. 462
<u>Relay 3</u> 1. -8 2. 53 3. -28 4. 5 5. 1000 6. 31	

Solutions:

Relay 1

1. The vertex for this parabola occurs at $x = -\frac{-3}{2 \cdot 2} = \frac{3}{4}$, so the y -coordinate is $y = 2\left(\frac{3}{4}\right)^2 - 3\left(\frac{3}{4}\right) - \frac{31}{8} = \frac{9}{8} - \frac{9}{4} - \frac{31}{8} = -5$.

2. $\left|(-5 + 12i)^2\right| = |-5 + 12i|^2 = \left(\sqrt{(-5)^2 + 12^2}\right)^2 = 25 + 144 = 169$

3. $\left(\sqrt{M} + \sqrt{168}\right)^2 = M + 2\sqrt{168M} + 168 = M + 4\sqrt{42M} + 168$, so because $42 = 7 \cdot 3 \cdot 2$, we need one more copy of each of these factors under the radical. Therefore, $M = 42$.

4. $\frac{x^3 - 12x^2 + 42x - 64}{x^2 + x - 5} = x - 13 + \frac{60x - 129}{x^2 + x - 5}$, so $g(x) = 60x - 129$. Therefore, $g(3) = 60 \cdot 3 - 129 = 180 - 129 = 51$.

5. Since the prime factorization of 51 is $3 \cdot 17$, the sum of the positive integral factors of 51 is $(1 + 3)(1 + 17) = 4 \cdot 18 = 72$.

6. Adding the corresponding sides to both of the first two equations yields $5x + y + 3z = 38$. Subtracting the corresponding sides of the third equation from this result yields $2z = 38 - 72 = -34 \Rightarrow z = -17$.

Relay 2

1. The total cost of the sandwiches is $72 \cdot \$3.79 = \272.88 . Therefore, according to the question, they paid \$273, meaning they received 12 cents as change.

2. The common difference d of this sequence satisfies the equation $d = \frac{395 - 12}{n - 1} = \frac{383}{n - 1}$, where

n is the number of terms in the sequence. Therefore, $n - 1 = \frac{383}{d}$, but since $n > 2$, d must be a proper divisor of 383. 383 is a prime number, so the only possible values of d is 1 (and thus the sequence has 384 terms. Therefore, the 75th term of the sequence is $12 + (75 - 1) \cdot 1 = 86$.

3. This inequality is equal to $n > \frac{2026 \cdot 1000}{T^2} = \frac{2,026,000}{86^2} \approx 273.9\dots$, so the least possible integer that works is 274.

4. Since $274^2 = 75,076$, the sum of the digits of 274^2 is $7 + 5 + 0 + 7 + 6 = 25$.

5. Since $(-2026, 2026)$ is outside the circle with equation $(x+2)^2 + (y-3)^2 = 25$, a line drawn through $(-2026, 2026)$ and the center of the circle will intersect the circle in the points nearest and farthest from $(-2026, 2026)$. Therefore, the difference in these distances is simply the length of the diameter of the circle, which is $2\sqrt{25} = 2 \cdot 5 = 10$.

6. Let n be the number of orange marbles in the bag. Therefore, $\frac{1}{3} = \frac{\binom{n}{2}}{\binom{10}{2}} = \frac{\frac{n(n-1)}{2 \cdot 1}}{\frac{10 \cdot 9}{2 \cdot 1}} = \frac{n(n-1)}{90}$

$\Rightarrow n(n-1) = 30 \Rightarrow n = 6$ (since $n > 0$). Therefore, there are 6 orange marbles in the bag, meaning there are 4 blue marbles in the bag.

Relay 3

1. $y = \frac{x^2 - 8x - 20}{x^2 + 6x + 8} = \frac{(x+2)(x-10)}{(x+2)(x+4)}$, so the hole is at $x = -2$, and the y -value of the hole is

$y = \frac{(-2) - 10}{(-2) + 4} = \frac{-12}{2} = -6$. Therefore, $a + b = (-2) + (-6) = -8$.

2. The equation we need to solve is $\cos(-8x) = -\frac{\sqrt{3}}{2}$, though because the cosine function is

even, we can simply solve $\cos(8x) = -\frac{\sqrt{3}}{2}$ instead. The least possible positive angle with a

cosine value of $-\frac{\sqrt{3}}{2}$ is $\frac{5\pi}{6}$, meaning the least possible solution to the equation satisfies $8x$

$= \frac{5\pi}{6} \Rightarrow x = \frac{5\pi}{48}$. Therefore, $M + N = 5 + 48 = 53$.

3. $C^2 + 2025 = C^2 + (-45)^2 = 53^2 = 2809 \Rightarrow C^2 = 784 \Rightarrow C = -28$ (since the point was in Quadrant III, $C < 0$)

4. For this hyperbola, the distance from the center to a vertex is $a = \left| \frac{7}{20} \right| = \left| \frac{-28}{20} \right| = \frac{7}{5}$. Therefore,

if c is the distance from the center to either focus of the hyperbola, we have $\frac{25}{7} = \frac{c}{\frac{7}{5}}$

$$\Rightarrow c = \frac{25}{7} \cdot \frac{7}{5} = 5.$$

5. In the expansion of $(5x^2 + 5x - 2)^5$, there are two ways to get an x^3 term:

$\frac{5!}{1!1!3!}(5x^2)^1(5x)^1(-2)^3$ and $\frac{5!}{0!3!2!}(5x^2)^0(5x)^3(-2)^2$. Therefore, the coefficient of x^3 would

be the sum of the coefficients of these two terms, which is $\frac{5!}{1!1!3!}(5)^1(5)^1(-2)^3$

$$+ \frac{5!}{0!3!2!}(5)^0(5)^3(-2)^2 = 20(5)(5)(-8) + 10(1)(125)(4) = -4000 + 5000 = 1000.$$

6. $\frac{1000}{57} = 17\frac{31}{57}$, so the remainder is 31.

Relay 4

1. The amount in the account is $\$10,000 \left(1 + \frac{.04}{2} \right)^{21} = \$10,000(1.02)^2 = \$10,000(1.0404) = \$10,404$.

$$2. \frac{10,404}{1+0+4+0+4} = \frac{10,404}{9} = 1156$$

3. Since $1156 = 2^2 \cdot 17^2$ is the prime factorization of 1156, $\ln 1156 = \ln(2^2 \cdot 17^2) = 2\ln 2 + 2\ln 17$, so $A + B + C + D = 2 + 2 + 2 + 17 = 23$.

4. If one of P and Q is positive while the other is negative, the resulting product will be negative; if both of P and Q are positive or both are negative, the resulting product will be positive, and

the resulting product would be the same in both cases, so we can just assume P and Q are both positive to achieve the greatest possible value. Therefore, $23 = \sqrt{P^2 + Q^2 + 3^2} \Rightarrow 529 = P^2 + Q^2 + 9 \Rightarrow 520 = P^2 + Q^2$. After some examination, there are only two sets of positive integers that fit this description: 22 and 6, and 18 and 14. Therefore, the greatest possible value of product PQ is $18 \cdot 14 = 252$.

$$5. \lim_{x \rightarrow 12} \left(\frac{x^2 + 9x - 252}{x^2 - 21x + 108} \right) = \lim_{x \rightarrow 12} \frac{\cancel{(x-12)}(x+21)}{\cancel{(x-12)}(x-9)} = \frac{12+21}{12-9} = \frac{33}{3} = 11$$

6. For the first line, solving for parameter r in the second equation yields $r = y - 10$, then plugging into the first equation yields $x = 3 - 5(y - 10) = 3 - 5y + 50 \Rightarrow x = 53 - 5y$. For the second line, solving for parameter s in the first equation yields $s = \frac{x}{6}$, then plugging into the second equation yields $y = -9 - 11\left(\frac{x}{6}\right) \Rightarrow 6y = -54 - 11x \Rightarrow 11x + 6y = -54$. Now plugging the first of these results, solved for x , into the second of these results yields $11(53 - 5y) + 6y = -54 \Rightarrow 583 - 55y + 6y = -54 \Rightarrow 637 = 49y \Rightarrow y = 13 \Rightarrow x = -12 \Rightarrow F + G = (-12) + 13 = 1$.

Relay 5

$$1. \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 45 + 84 + 96 - 105 - 48 - 72 = 0$$

2. The only way for a perfect square to end in a 0 is if the number of which it is the square also ends in 0. To get a positive result less than 10,000, then, is to square 10, 20, 30, 40, 50, 60, 70, 80, or 90, so there are 9 such perfect squares.

$$3. \text{Simplifying, then using Vieta's formulas, } \frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}{\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}} = \frac{\frac{ab+ac+bc}{abc}}{\frac{a+b+c}{abc}} = \frac{ab+ac+bc}{a+b+c}$$

$$= \frac{\frac{595}{-2}}{\frac{17}{-2}} = -35.$$

4. This equation is $-4^x + 2^{x+6} - 1050 = -35 \Rightarrow 0 = 4^x - 2^{x+6} + 1015 = (2^x)^2 - 64 \cdot 2^x + 1015$
 $= (2^x - 29)(2^x - 35) \Rightarrow 2^x = 29$ or $2^x = 35 \Rightarrow x = \log_2 29$ or $x = \log_2 35$. Therefore, the sum of the solutions is $\log_2 29 + \log_2 35 = \log_2 (29 \cdot 35) = \log_2 1015$. Since $y = \log_2 x$ is an increasing function, $9 = \log_2 512 < \log_2 1015 < \log_2 1024 = 10$, so 10 is the least integer greater than or equal to the sum.

5. These numbers are of the form $\binom{12}{n}$, so the greatest value, based on the fact that numbers

in a row of Pascal's triangle increase until the middle, then decrease symmetrically, is $\binom{12}{6}$

$$= \frac{12!}{6!6!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 924.$$

6. For a logistic growth model, the rate of growth is always fastest at half of the carrying capacity, which for this model is $T = 924$. Therefore, the growth rate is fastest when $y = 462$.